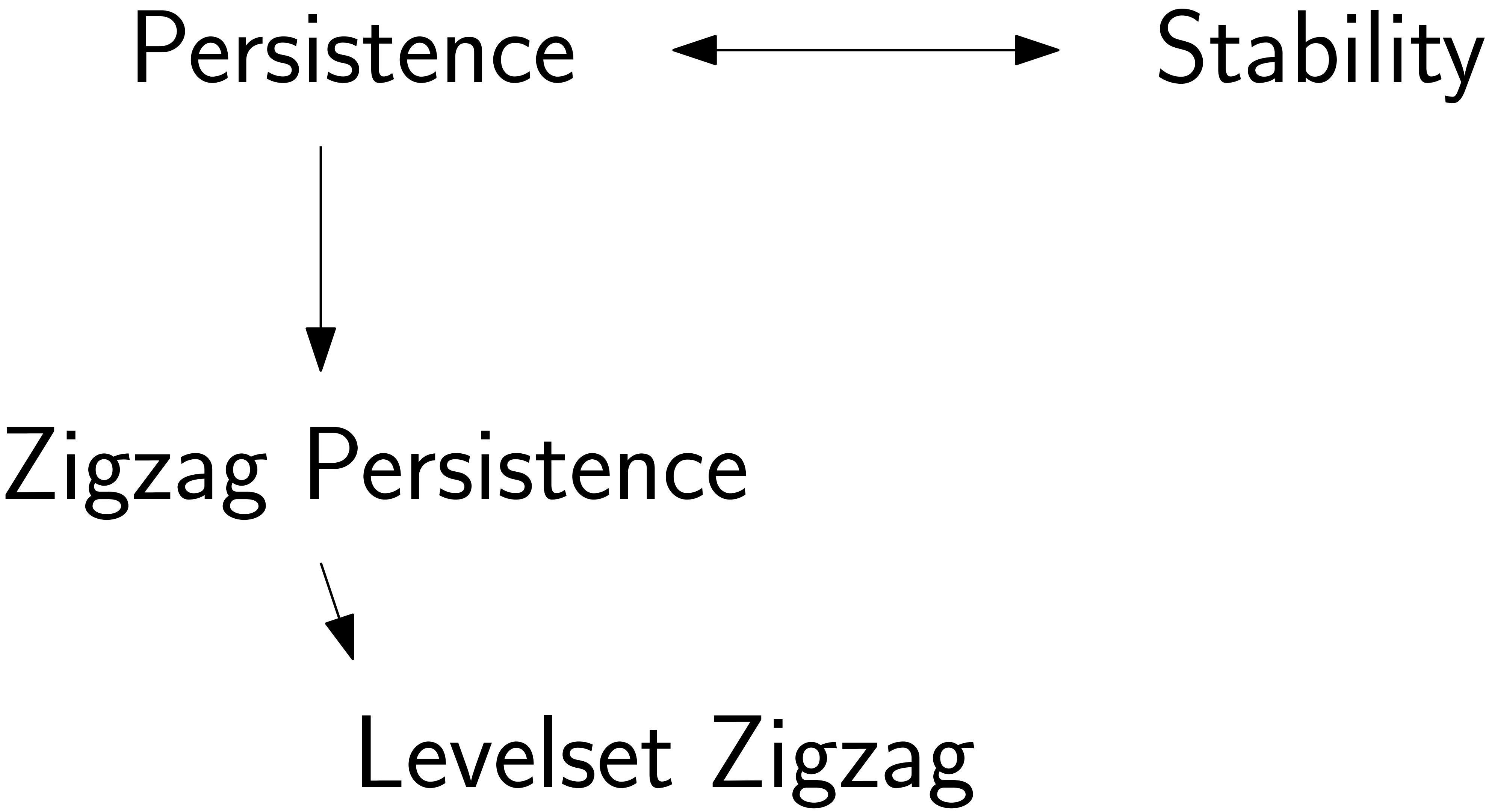
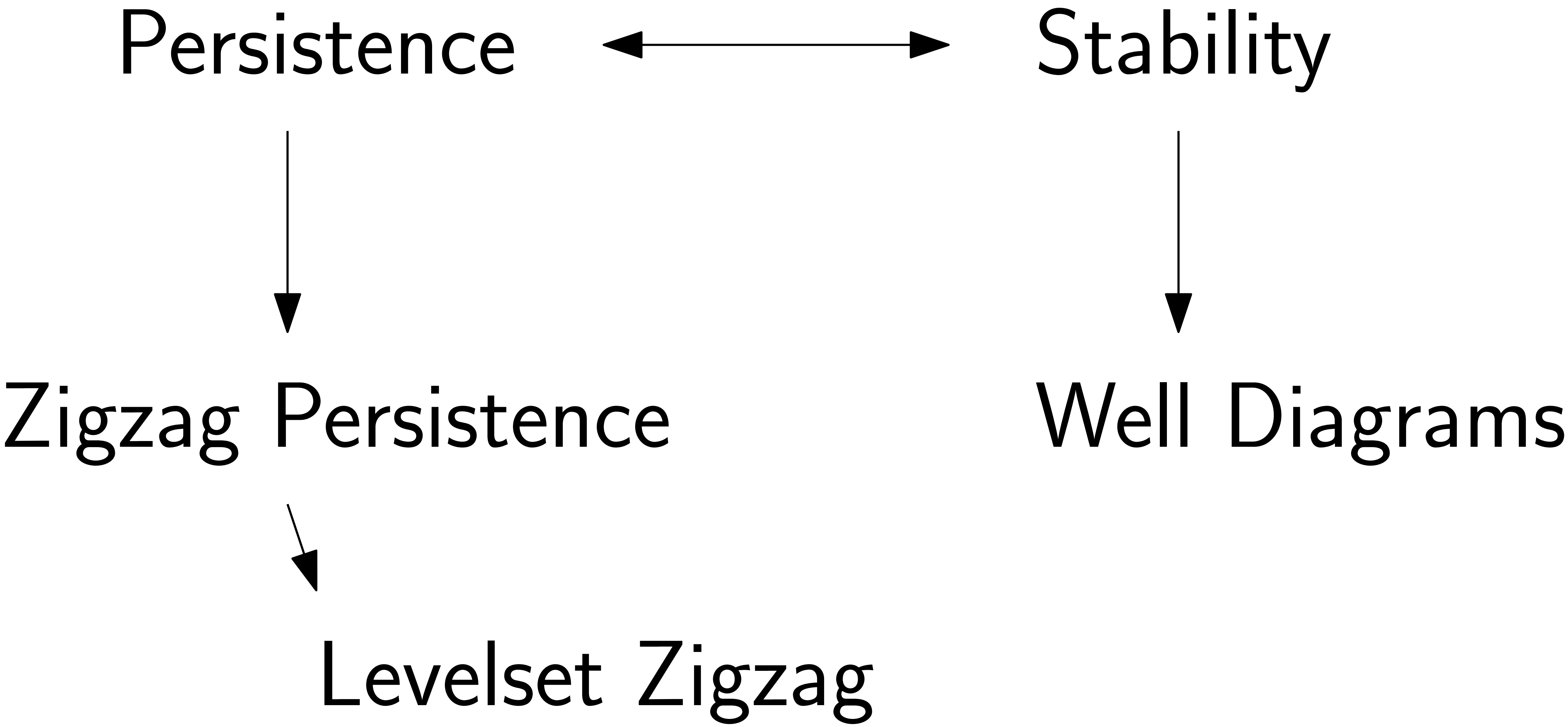
From the Levelset Zigzag to the Well Diagrams

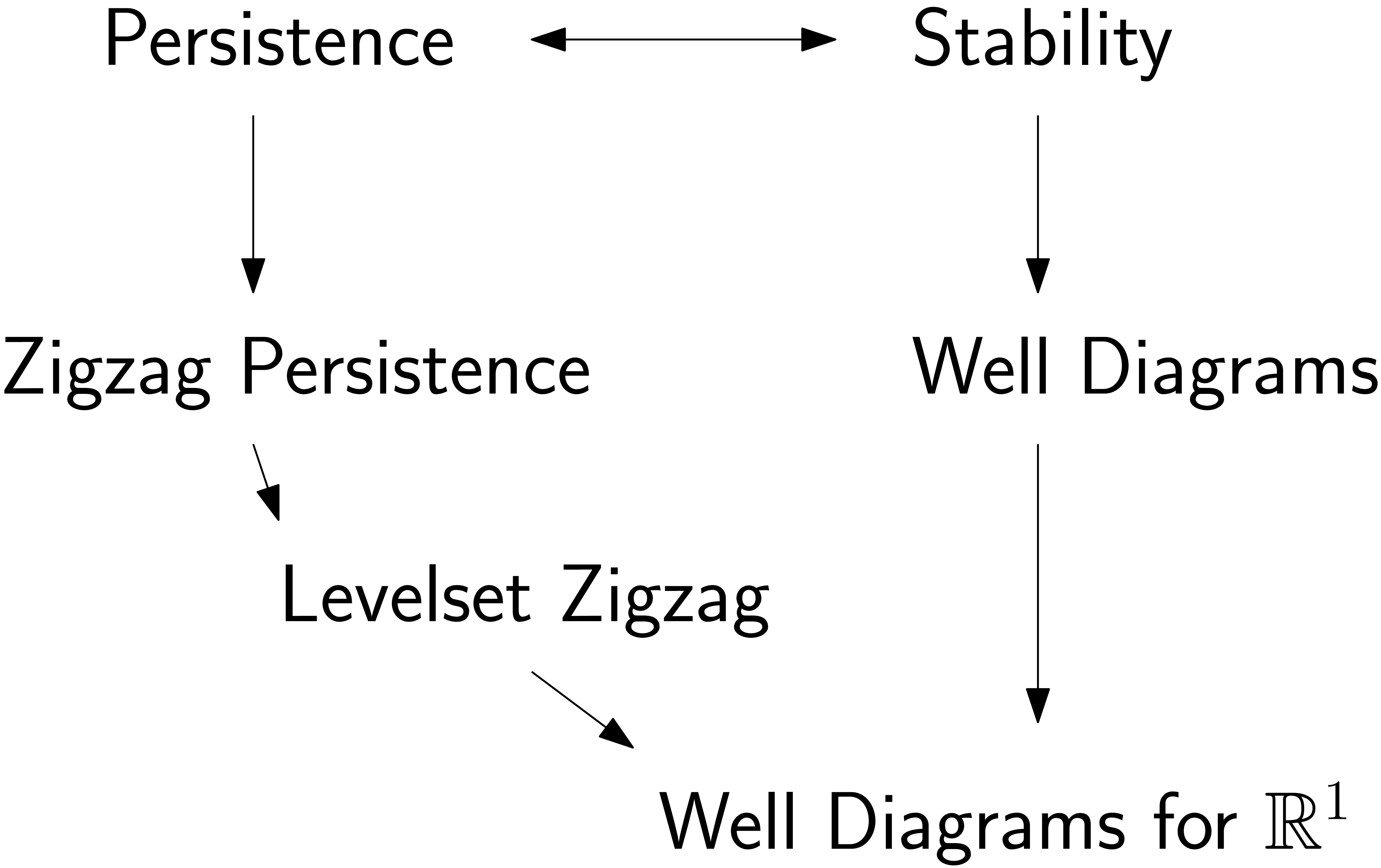
Dmitriy Morozov

Joint work with Gunnar Carlsson and Vin de Silva;
Paul Bendich, Herbert Edelsbrunner, and Amit Patel.

Persistence ⟷ Stability


Persistence
Stability
Zigzag Persistence
Levelset Zigzag


Persistence
Stability
Zigzag Persistence
Well Diagrams
Levelset Zigzag


Persistence
Stability
Zigzag Persistence
Well Diagrams
Levelset Zigzag
Well Diagrams for $\mathbb{R}^1$

Persistence

$$V_1 \to V_2 \to V_3 \to V_4 \to V_5 \to \ldots$$

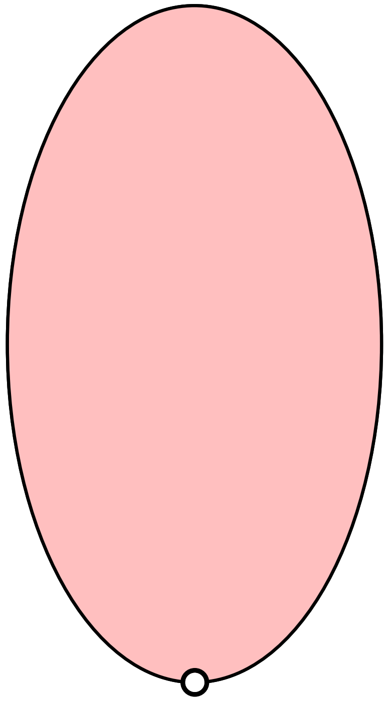
V_i = vector space over k (e.g. $\mathbb{Z}_2$)
(e.g. homology)

Persistence

$V_1 \to V_2 \to V_3 \to V_4 \to V_5 \to \dots$ $\quad V_i =$ vector space over k (e.g. $\mathbb{Z}_2$)

(e.g. homology)

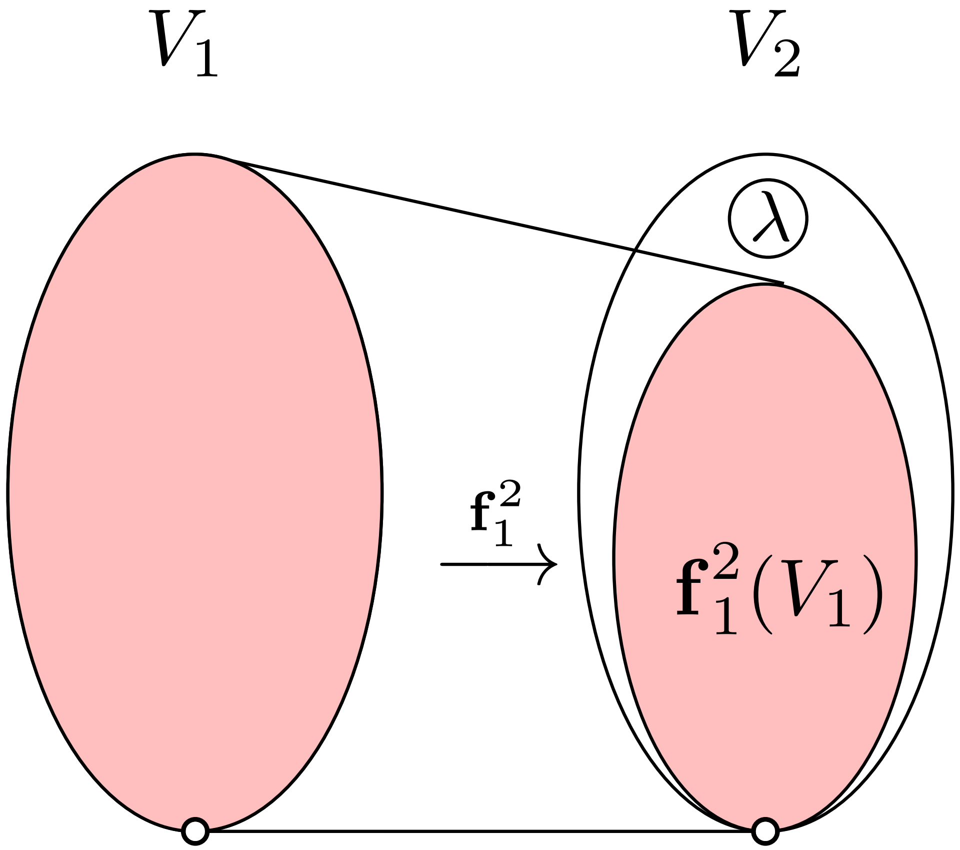
V_1



Persistence

$$V_1 \to V_2 \to V_3 \to V_4 \to V_5 \to \ldots$$

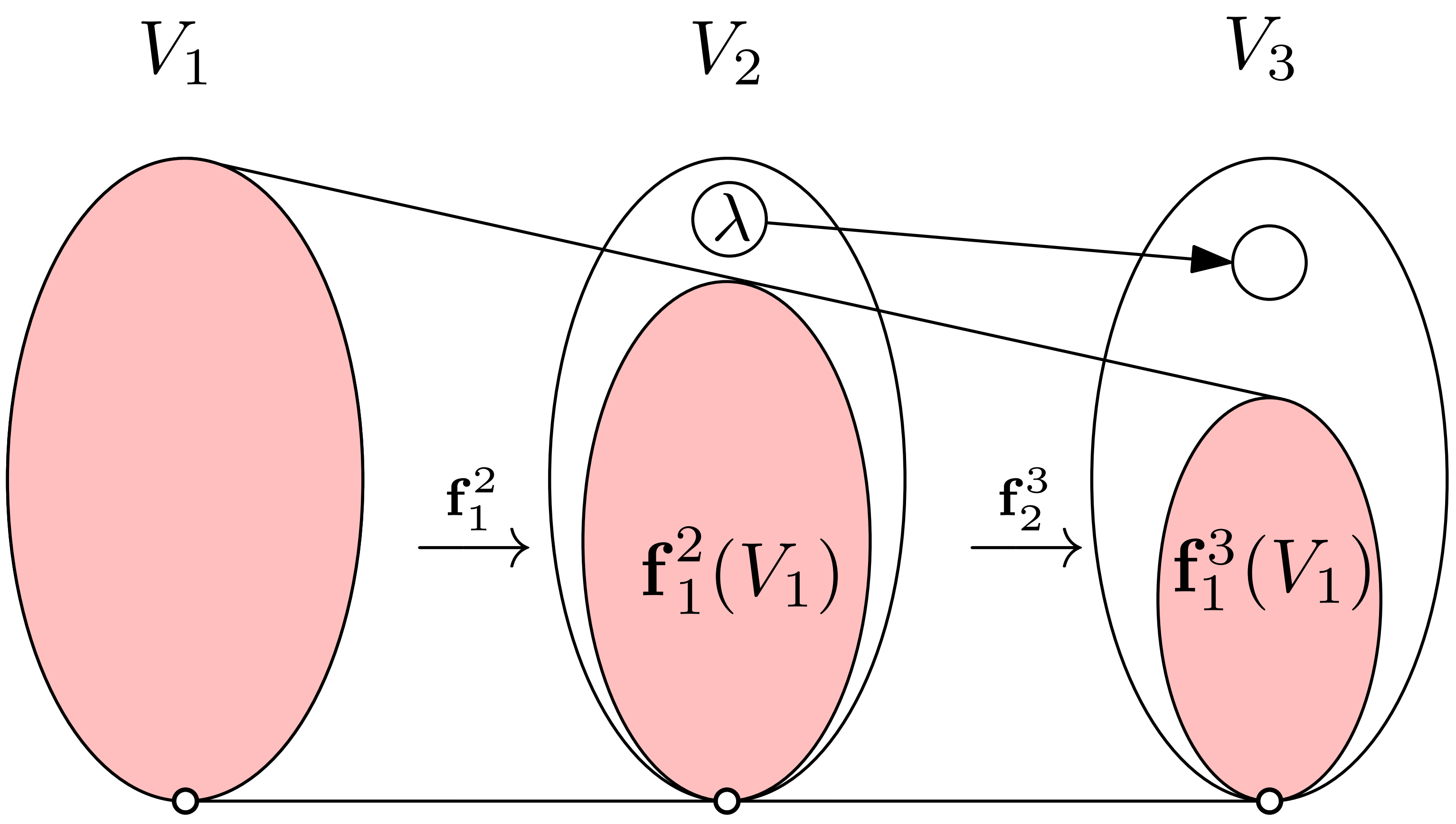
V_i = vector space over k (e.g. $\mathbb{Z}_2$)
(e.g. homology)



Persistence

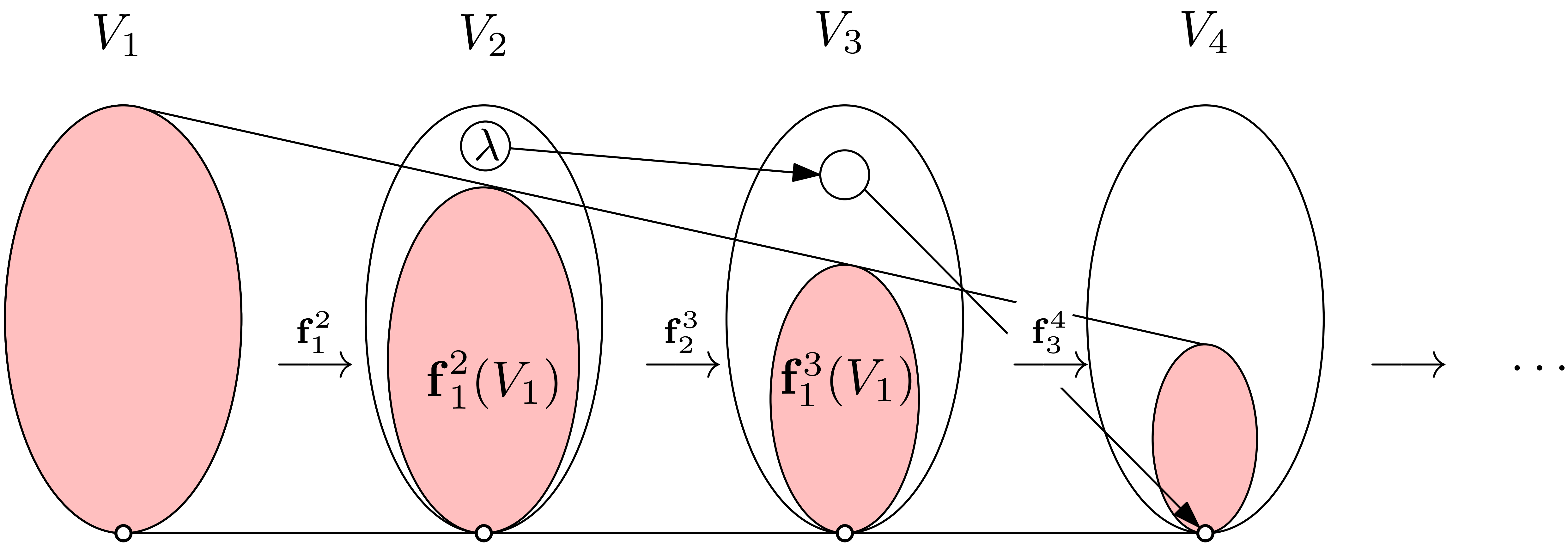
$$V_1 \to V_2 \to V_3 \to V_4 \to V_5 \to \dots$$

V_i = vector space over k (e.g. $\mathbb{Z}_2$)
(e.g. homology)



Persistence

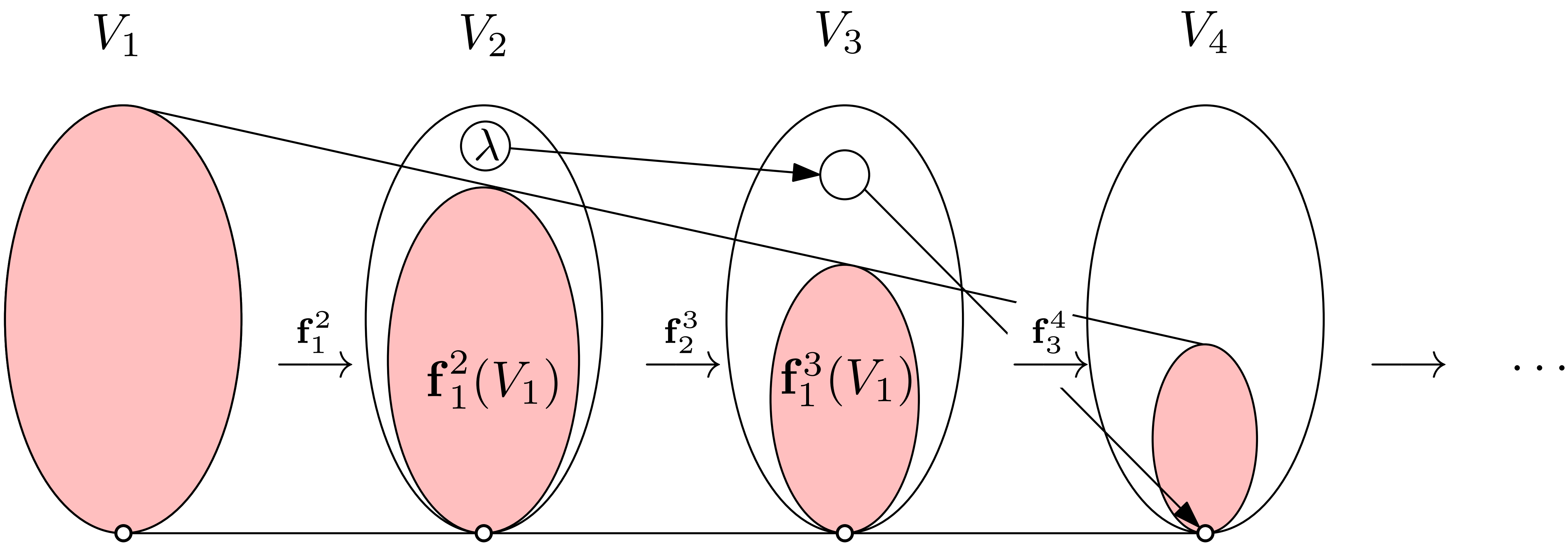
$V_1 \to V_2 \to V_3 \to V_4 \to V_5 \to \ldots$ $V_i =$ vector space over k (e.g. $\mathbb{Z}_2$)
(e.g. homology)



Persistence

$$V_1 \to V_2 \to V_3 \to V_4 \to V_5 \to \dots$$

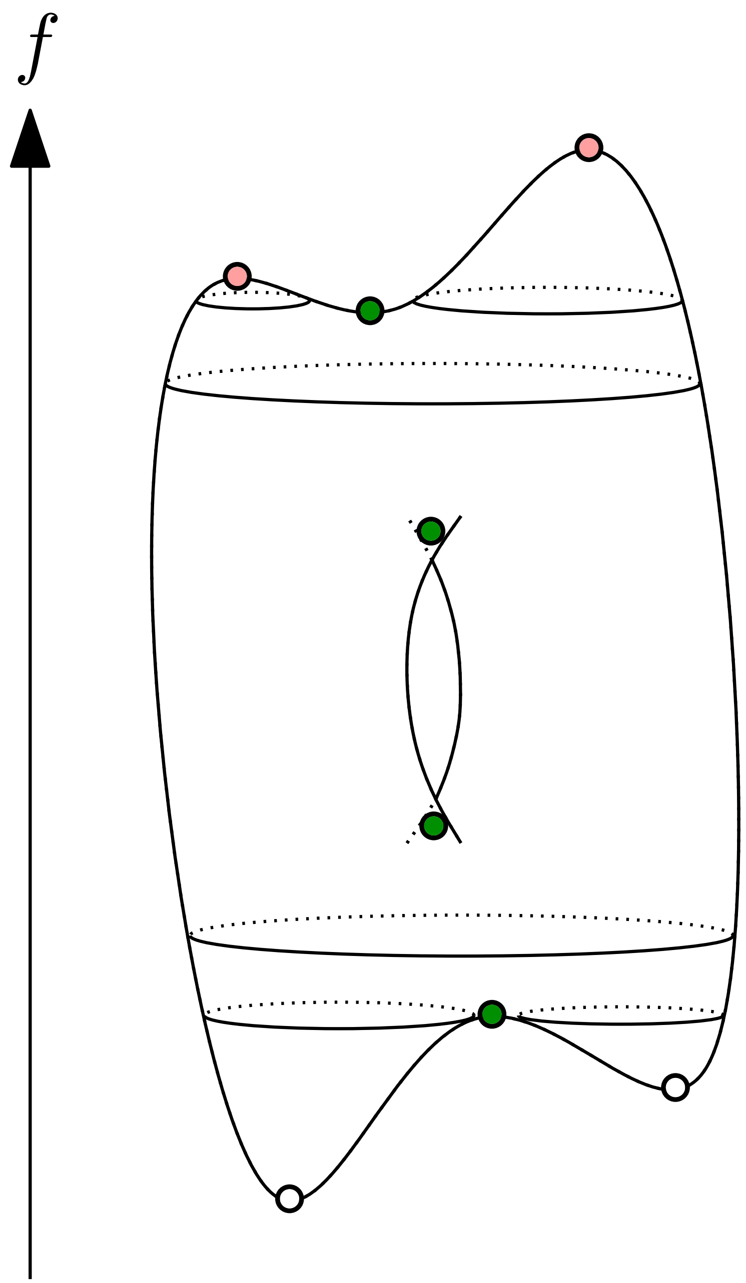
V_i = vector space over k (e.g. $\mathbb{Z}_2$)
(e.g. homology)

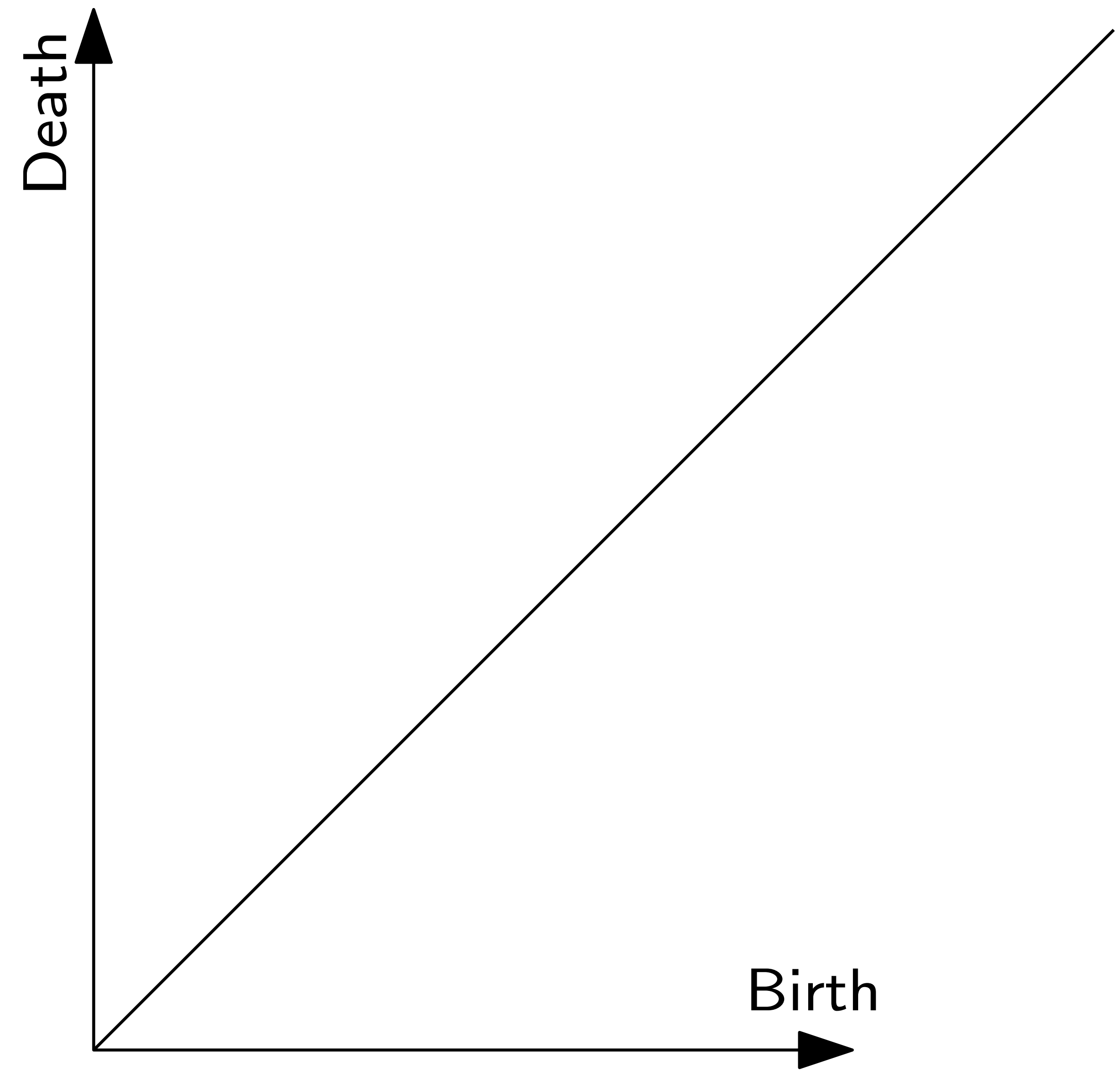


$$\begin{array}{ccccccc} & & k & \xrightarrow{1} & k & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ V_1 & \xrightarrow{\mathbf{f}_1^2} & \mathbf{f}_1^2(V_1) & \xrightarrow{\mathbf{f}_2^3} & \mathbf{f}_1^3(V_1) & \xrightarrow{\mathbf{f}_3^4} & \mathbf{f}_1^4(V_1) \end{array}$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]

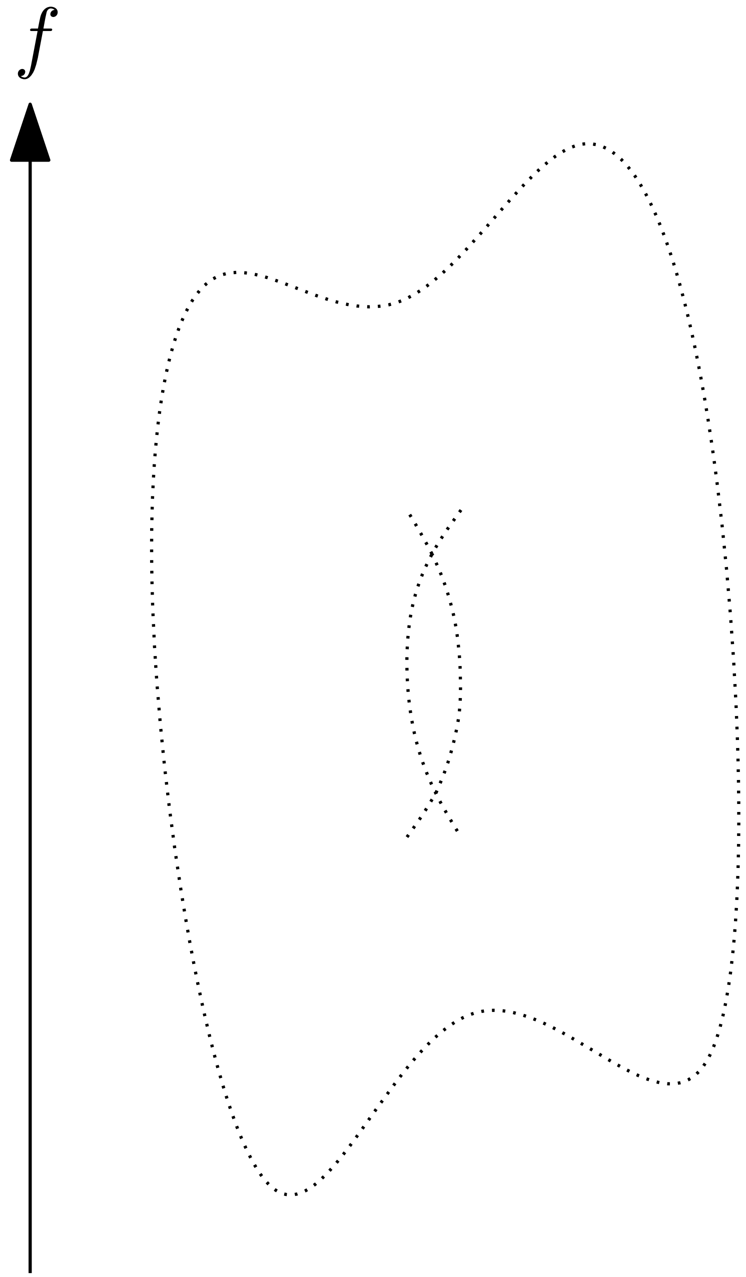


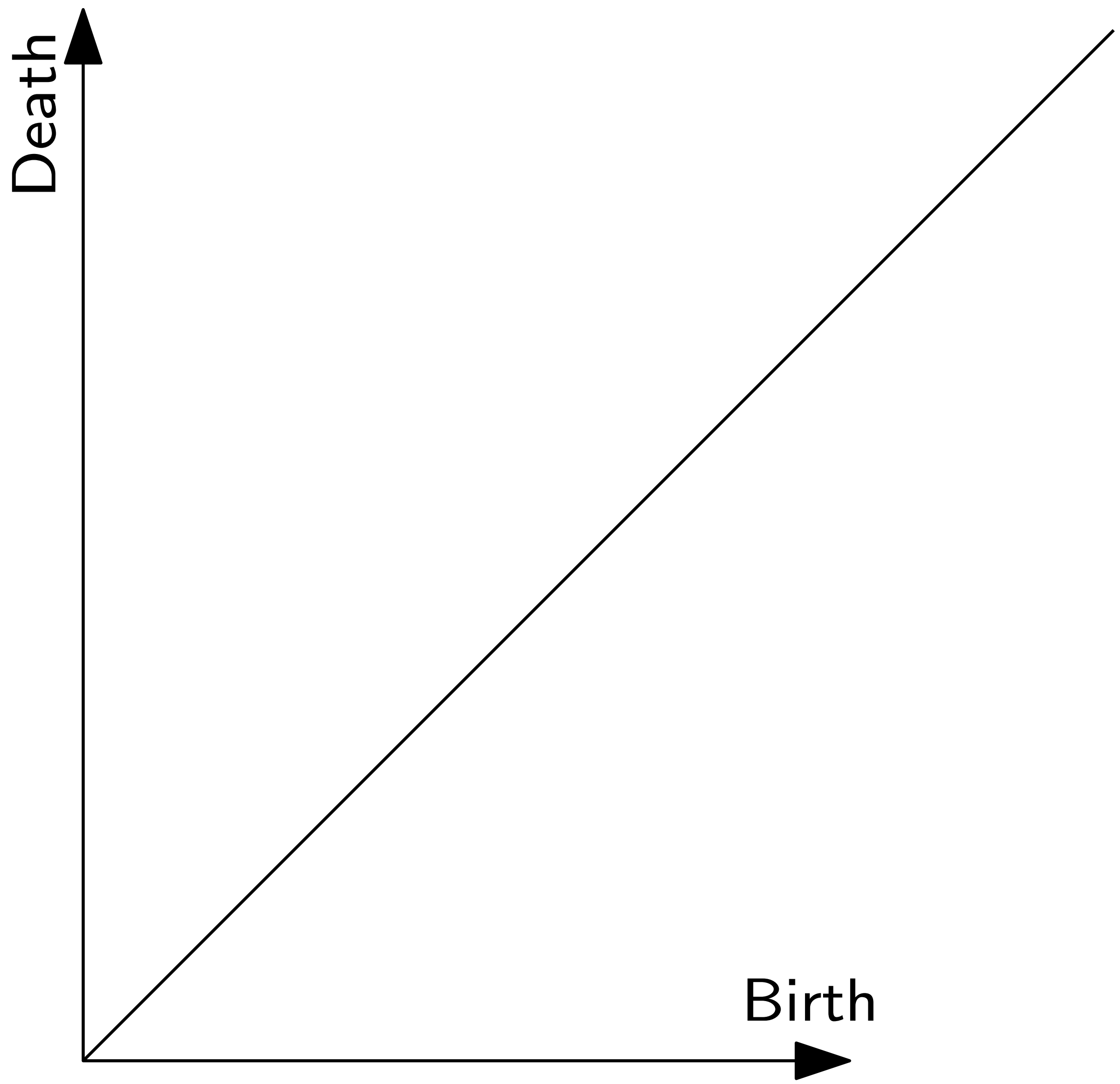


$$\begin{array}{ccccccc} \mathsf{H}(\mathbb{X}_1) & \rightarrow & \mathsf{H}(\mathbb{X}_2) & \rightarrow & \dots & \rightarrow & \mathsf{H}(\mathbb{X}) \\ & & & & & & \downarrow \\ \mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset) \end{array}$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]

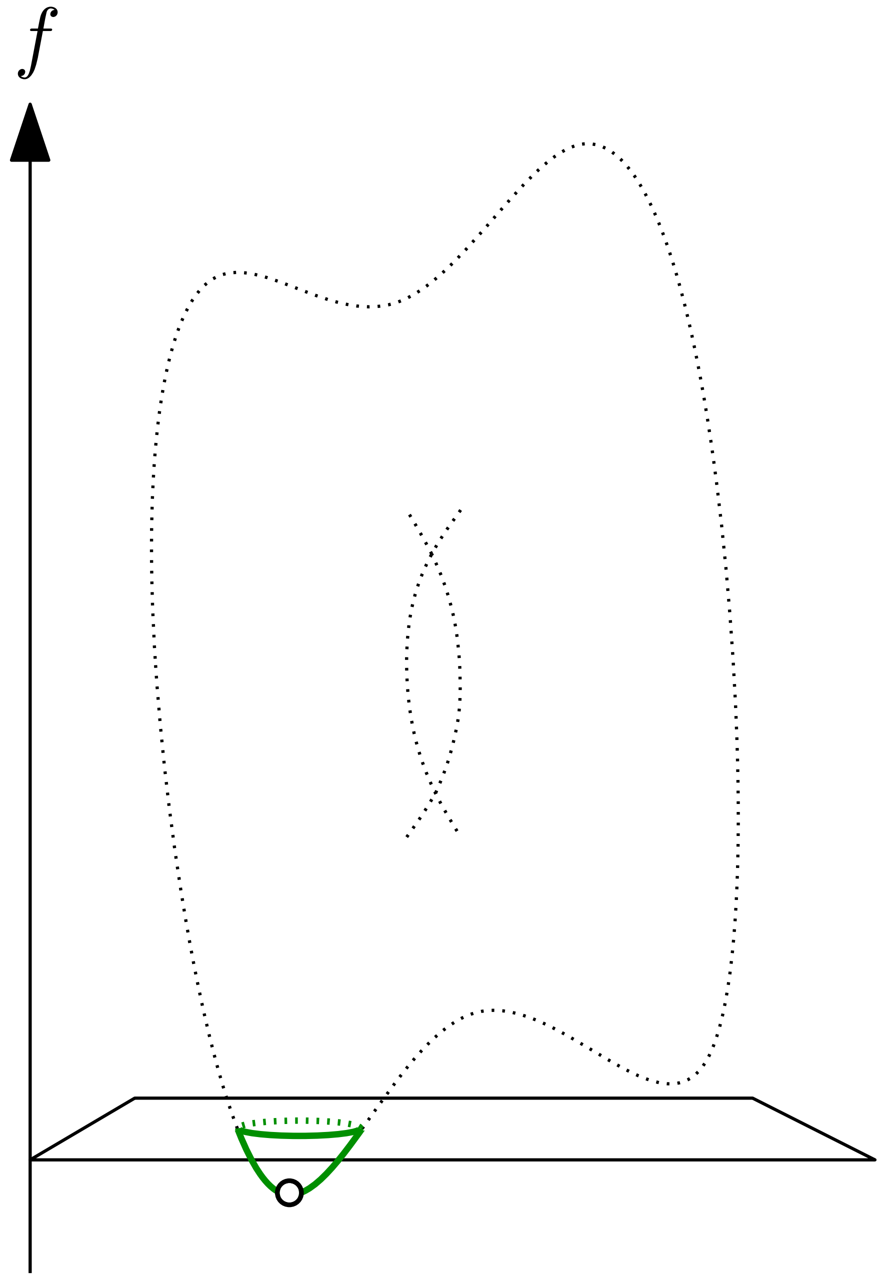


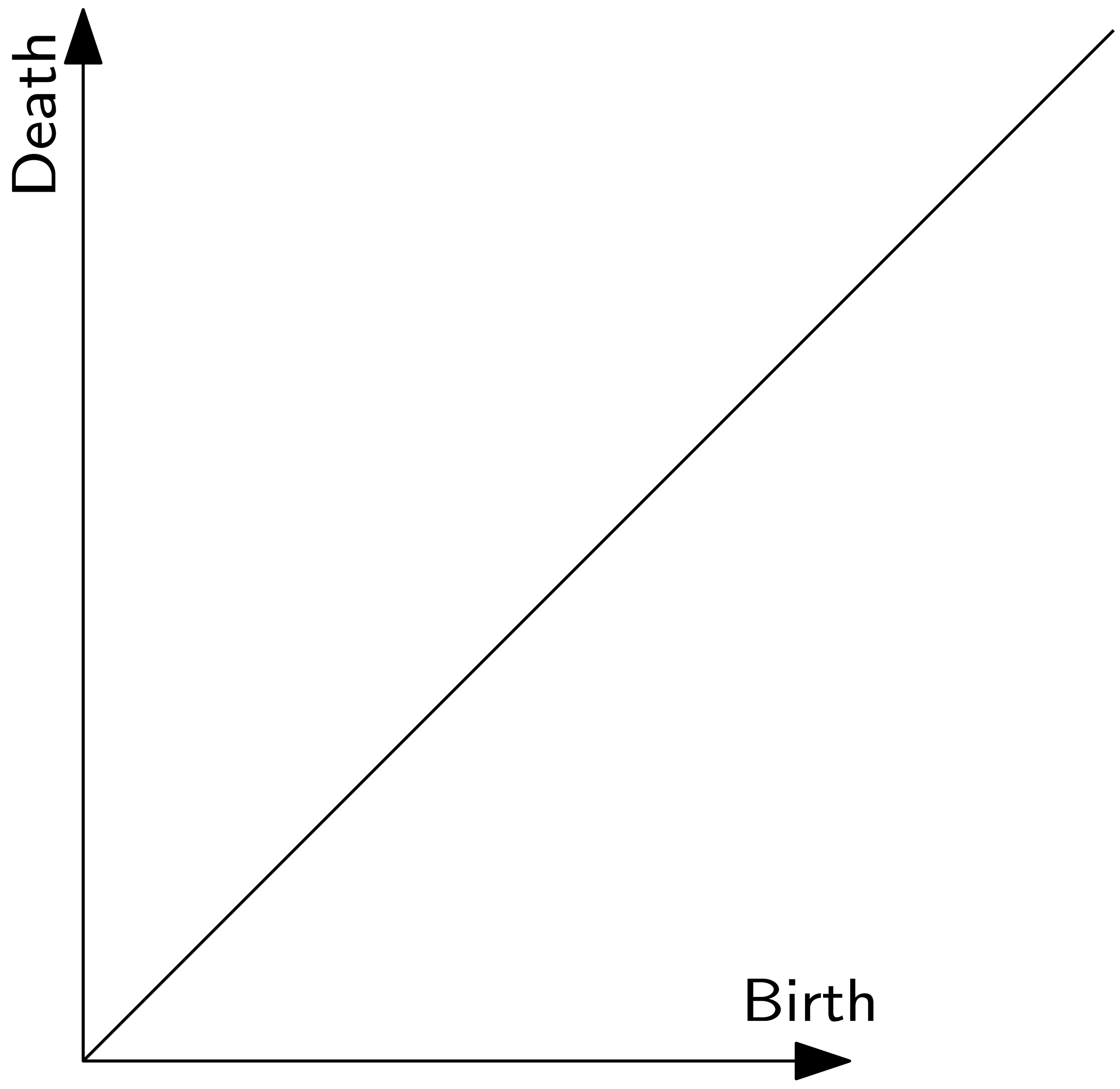


$$\begin{array}{ccccccc} \mathsf{H}(\mathbb{X}_1) & \rightarrow & \mathsf{H}(\mathbb{X}_2) & \rightarrow & \dots & \rightarrow & \mathsf{H}(\mathbb{X}) \\ & & & & & & \downarrow \\ \mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset) \end{array}$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]

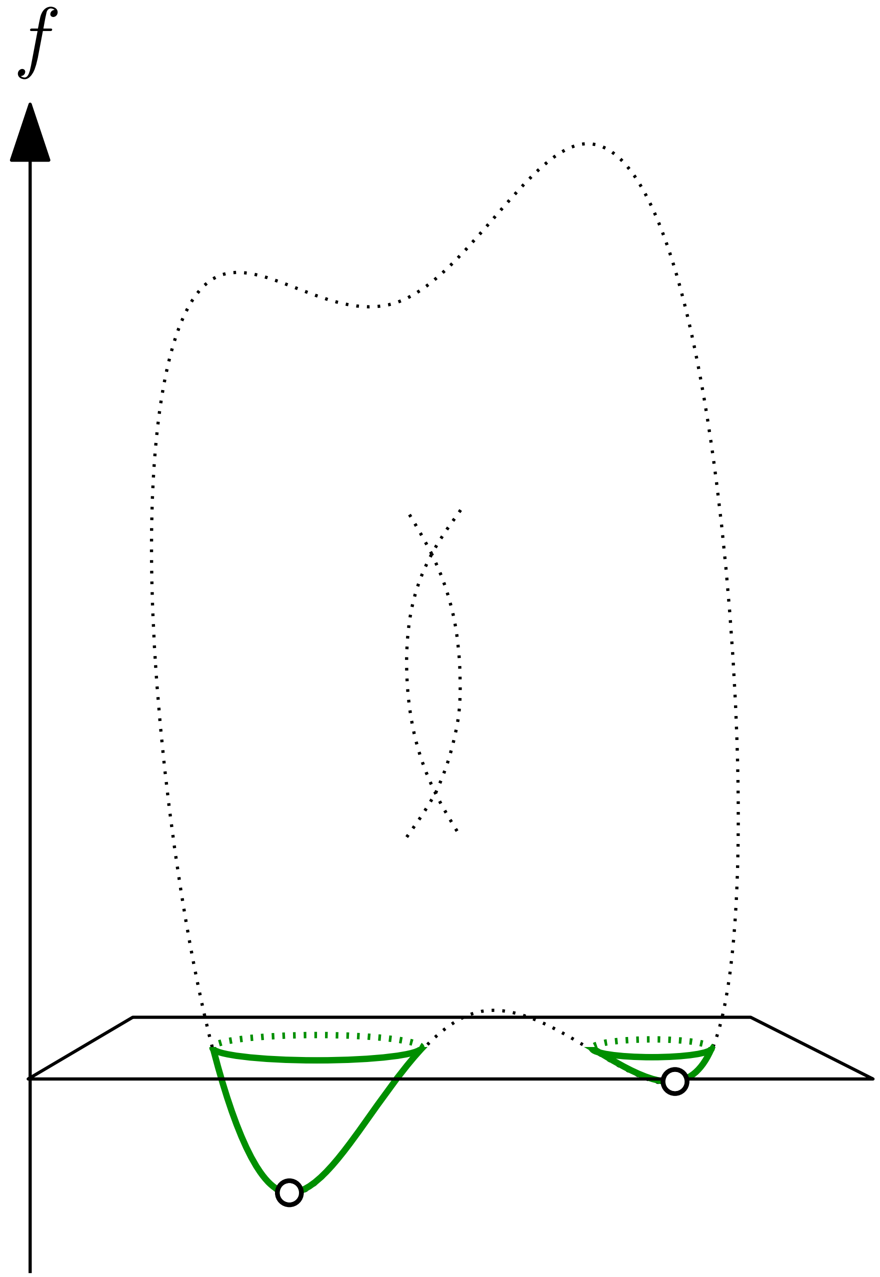


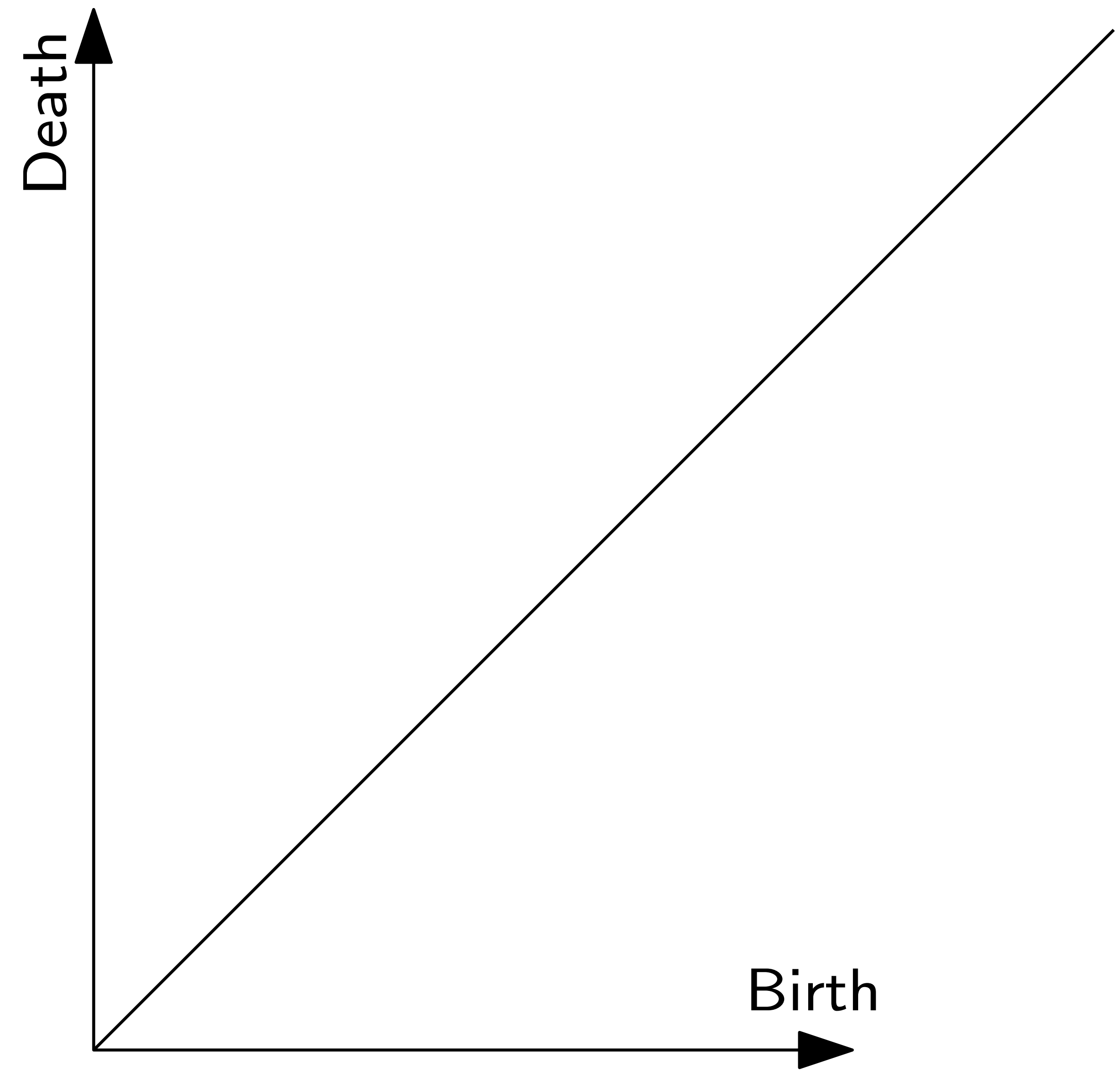


$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \rightarrow & \mathsf{H}(\mathbb{X}_2) & \rightarrow & \ldots & \rightarrow & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \ldots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]

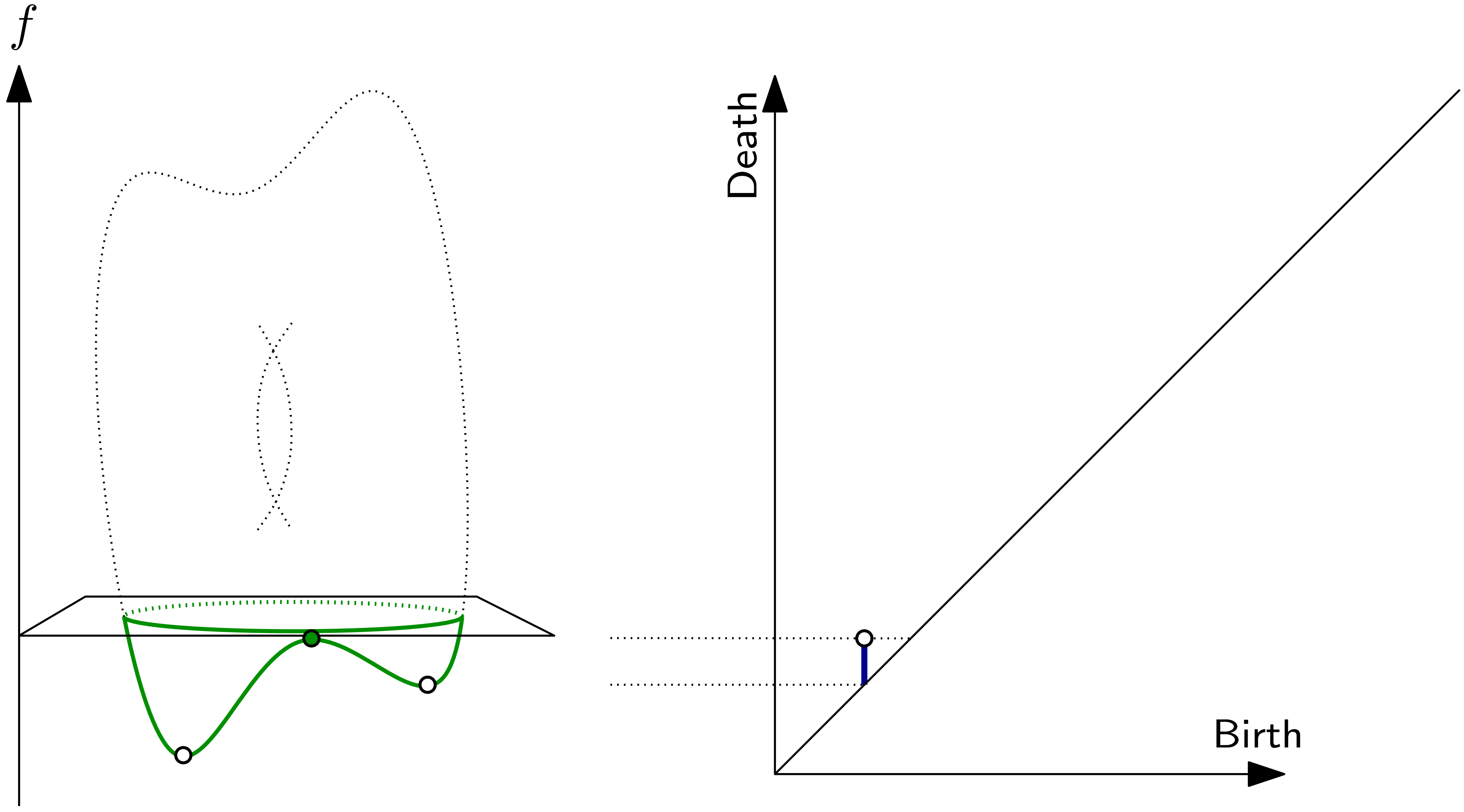




$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]



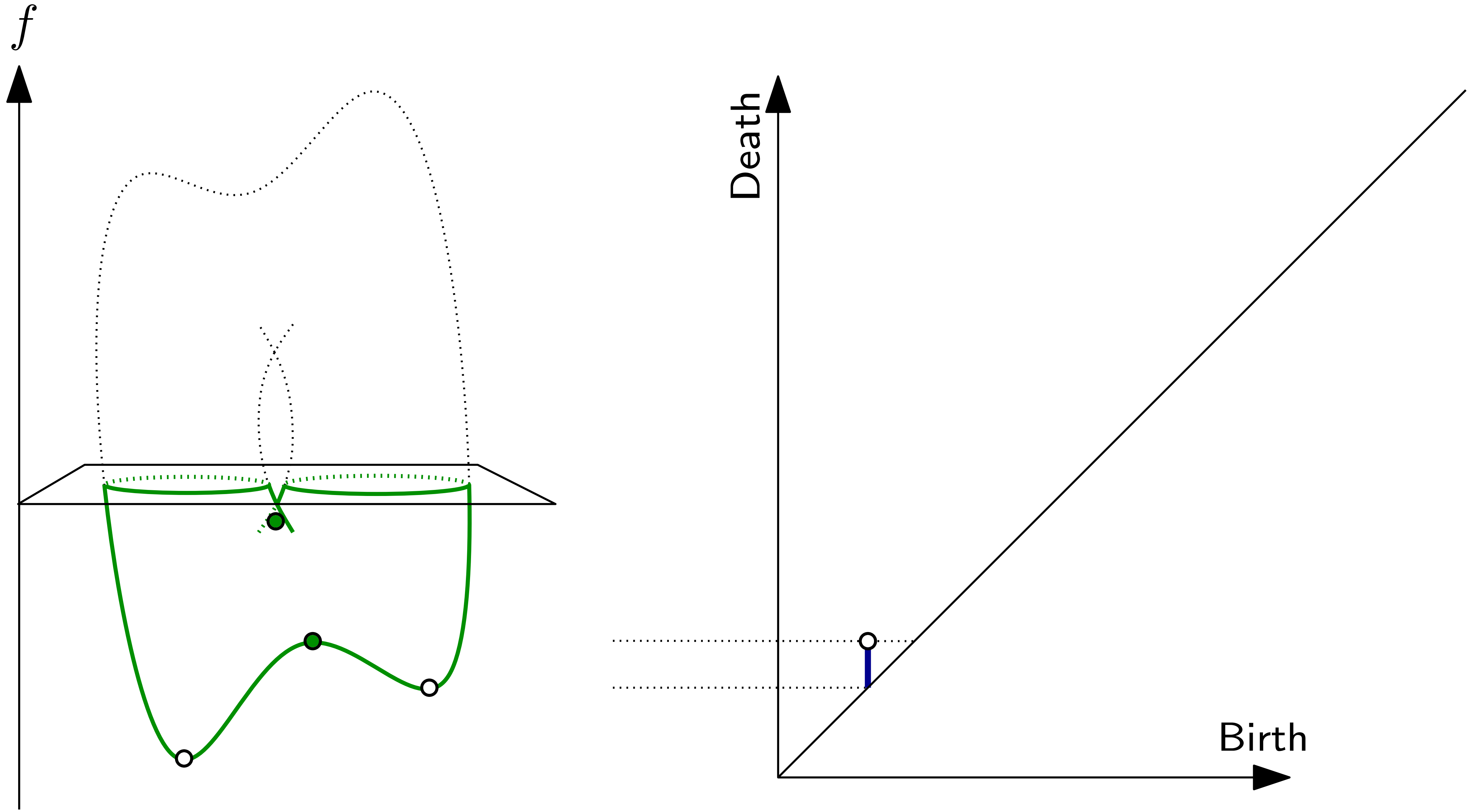
$$\mathsf{H}(\mathbb{X}_1) \rightarrow \mathsf{H}(\mathbb{X}_2) \rightarrow \dots \rightarrow \mathsf{H}(\mathbb{X})$$

$$\downarrow$$

$$\mathsf{H}(\mathbb{X}, \mathbb{X}^1) \leftarrow \dots \leftarrow \mathsf{H}(\mathbb{X}, \mathbb{X}^n) \leftarrow \mathsf{H}(\mathbb{X}, \emptyset)$$

Extended Persistence

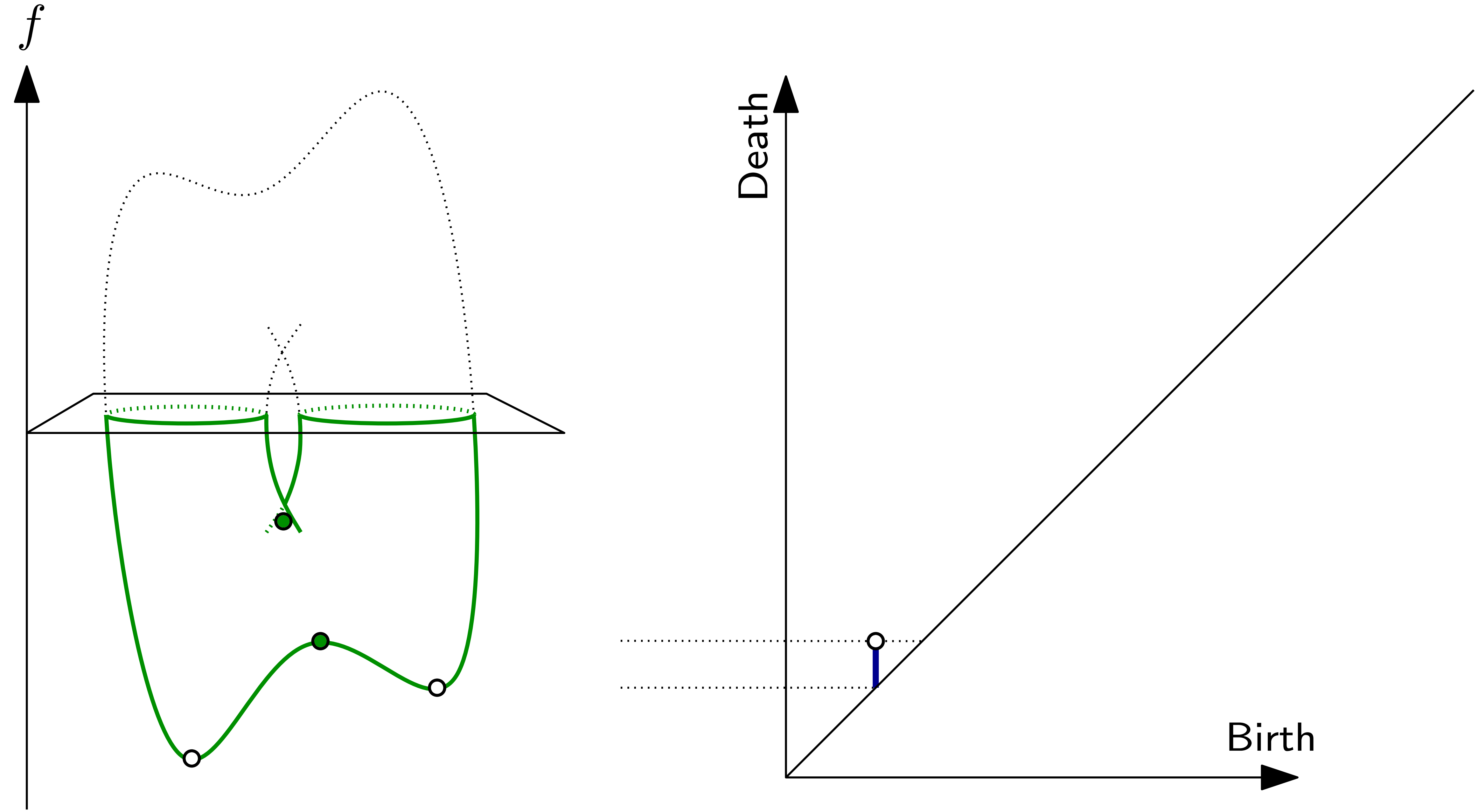
[Cohen-Steiner, Edelsbrunner, Harer '05]



$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \rightarrow & \mathsf{H}(\mathbb{X}_2) & \rightarrow & \dots & \rightarrow & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

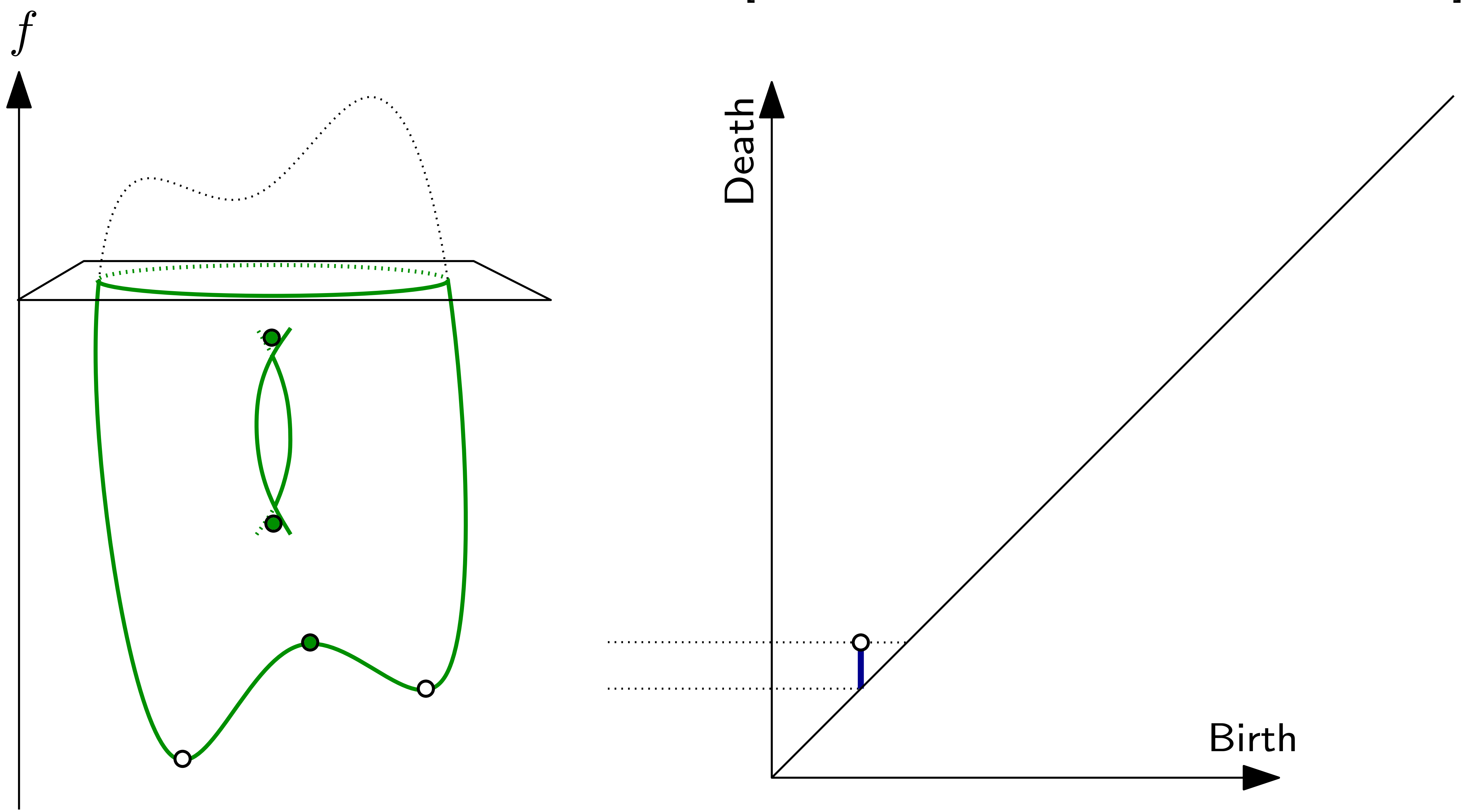
[Cohen-Steiner, Edelsbrunner, Harer '05]



$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \rightarrow & \mathsf{H}(\mathbb{X}_2) & \rightarrow & \dots & \rightarrow & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

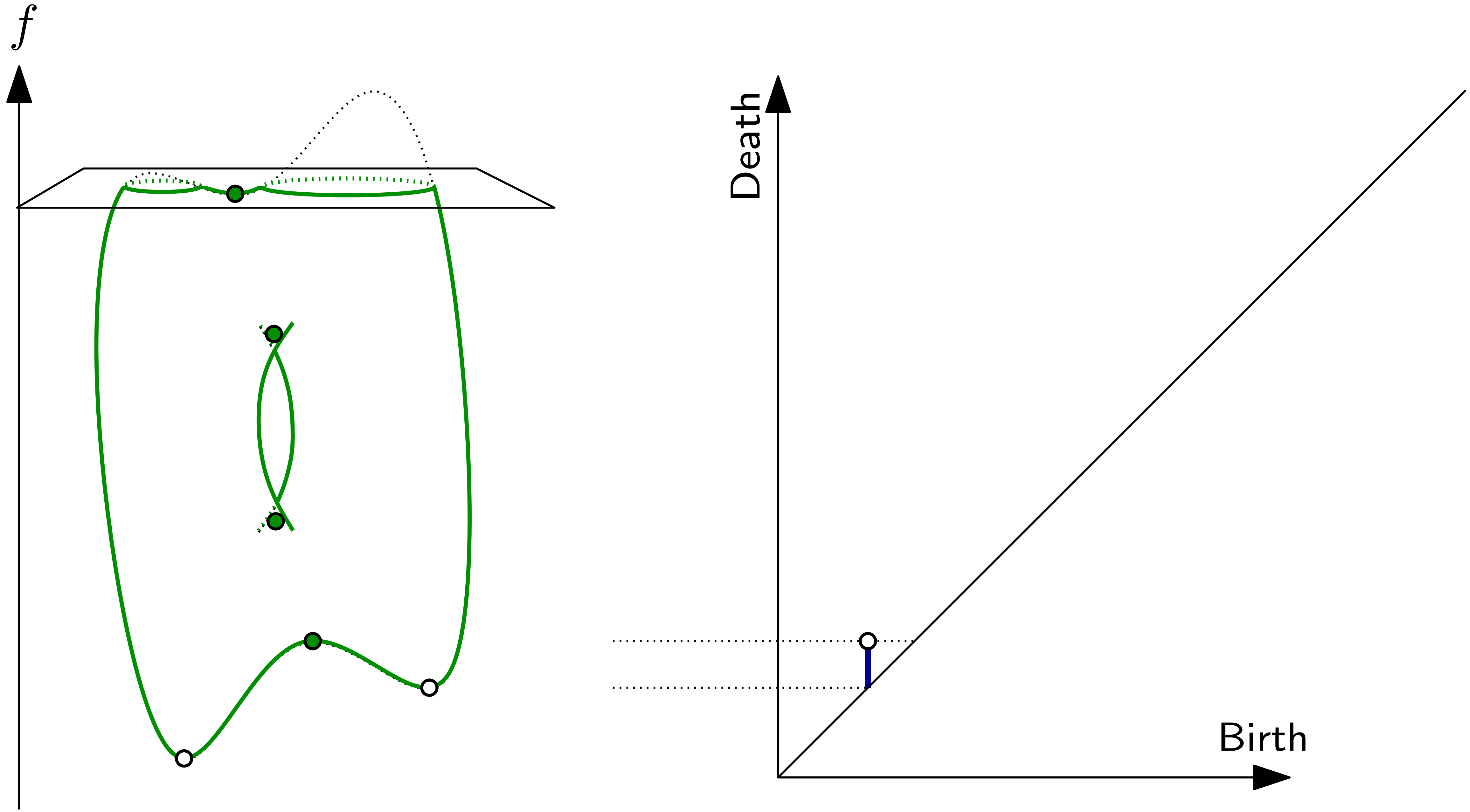
[Cohen-Steiner, Edelsbrunner, Harer '05]



$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

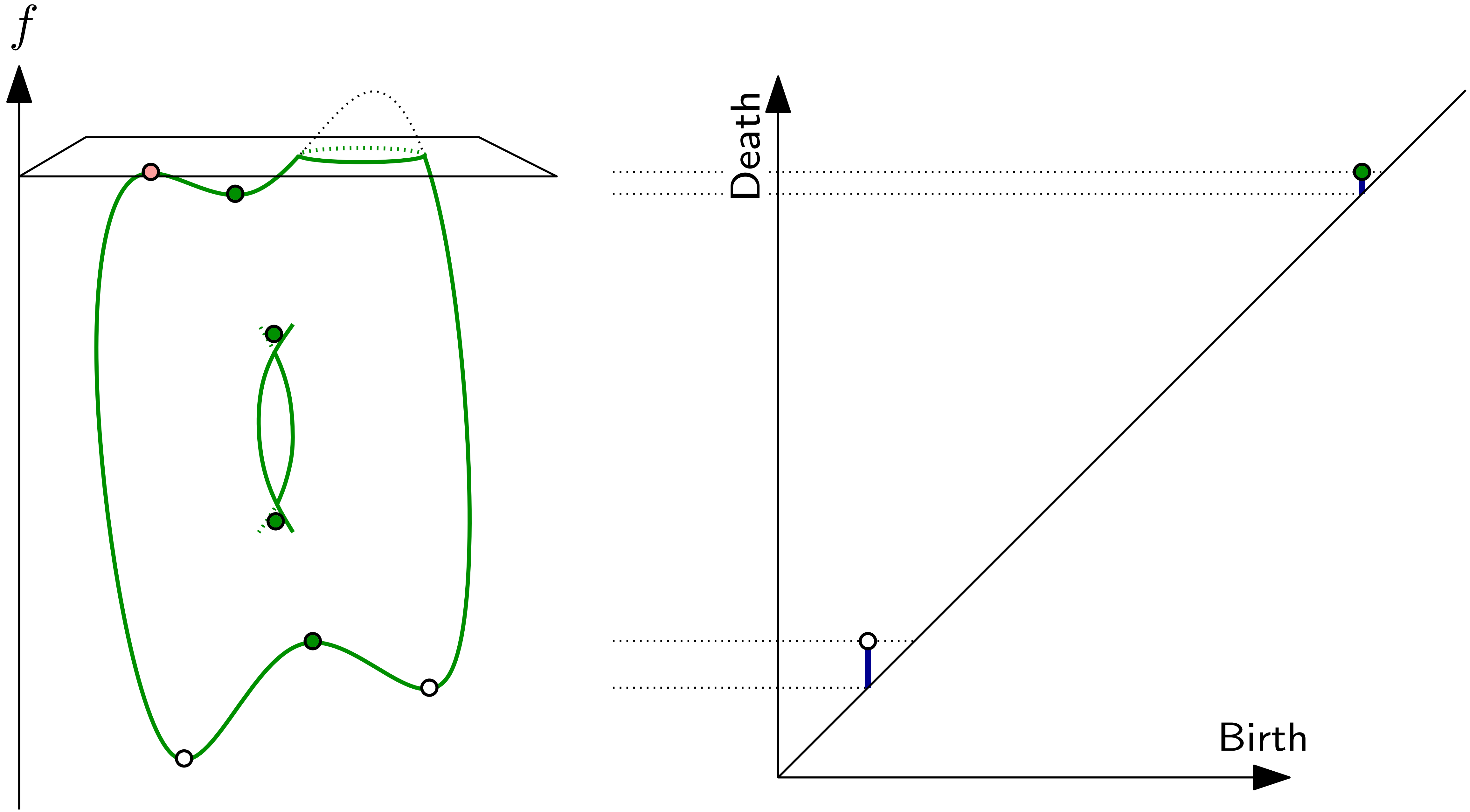
[Cohen-Steiner, Edelsbrunner, Harer '05]



$$\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \rightarrow & \mathsf{H}(\mathbb{X}_2) & \rightarrow & \dots & \rightarrow & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}$$

Extended Persistence

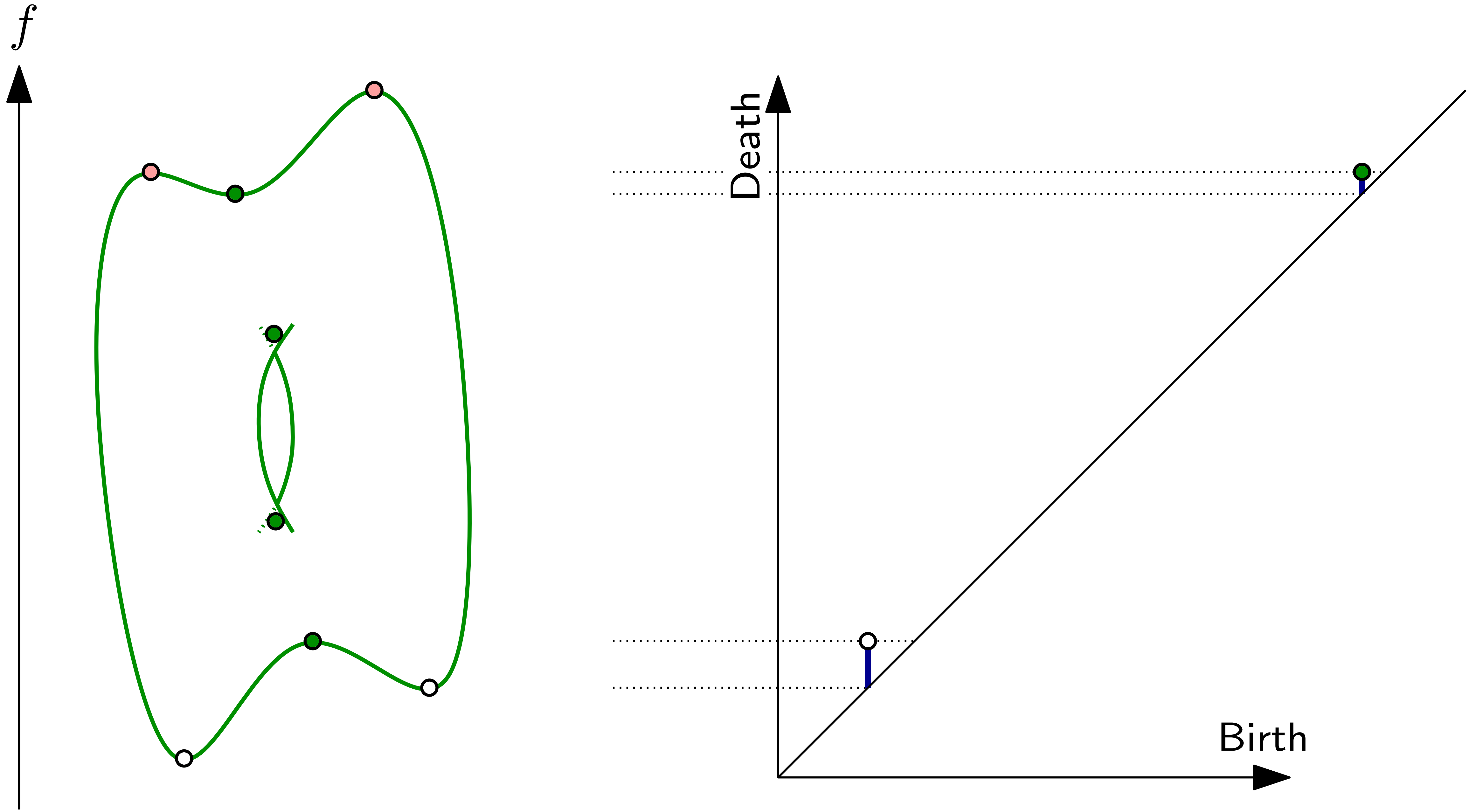
[Cohen-Steiner, Edelsbrunner, Harer '05]



$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \rightarrow & \mathsf{H}(\mathbb{X}_2) & \rightarrow & \ldots & \rightarrow & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \ldots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

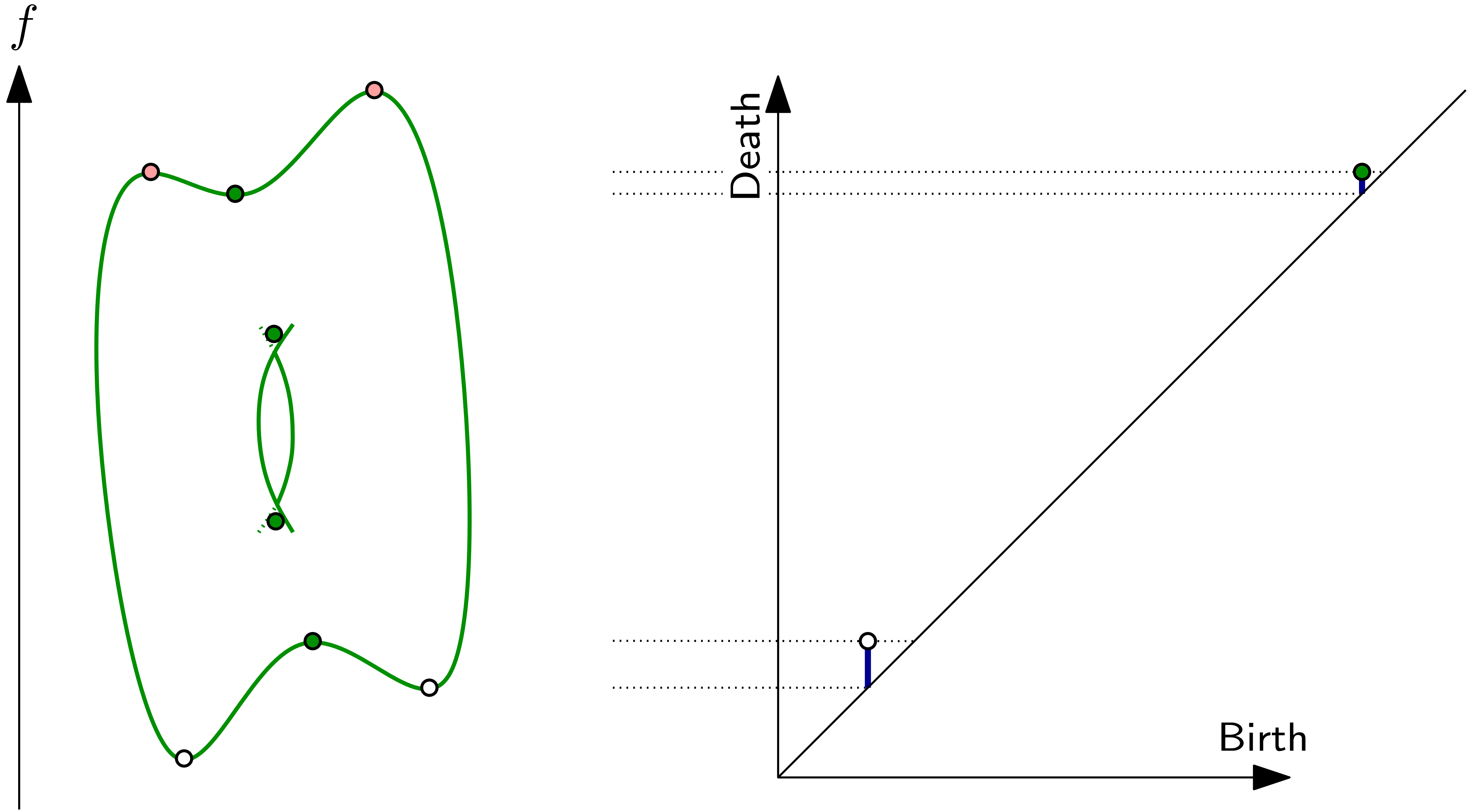
[Cohen-Steiner, Edelsbrunner, Harer '05]



$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \rightarrow & \mathsf{H}(\mathbb{X}_2) & \rightarrow & \dots & \rightarrow & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

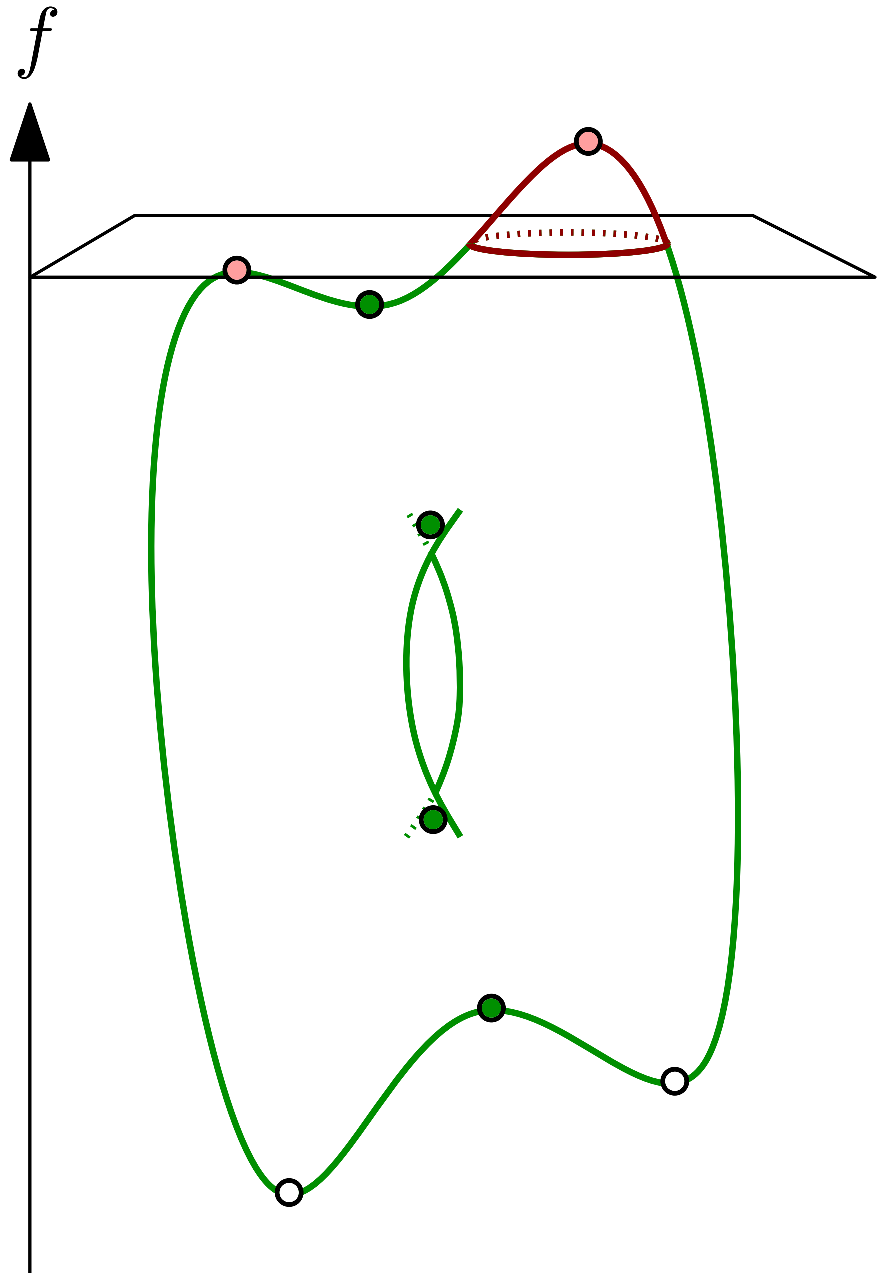
[Cohen-Steiner, Edelsbrunner, Harer '05]

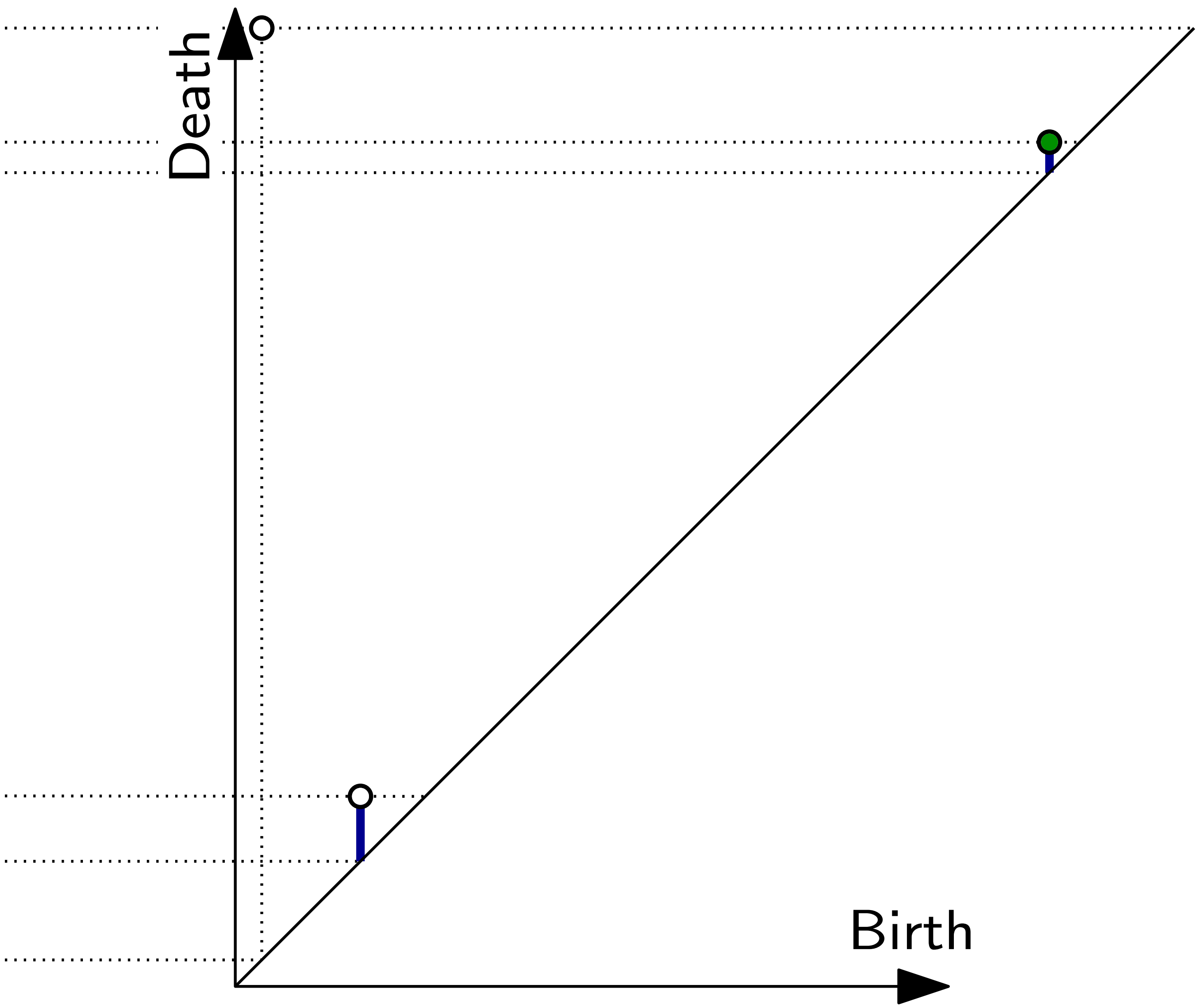


$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]

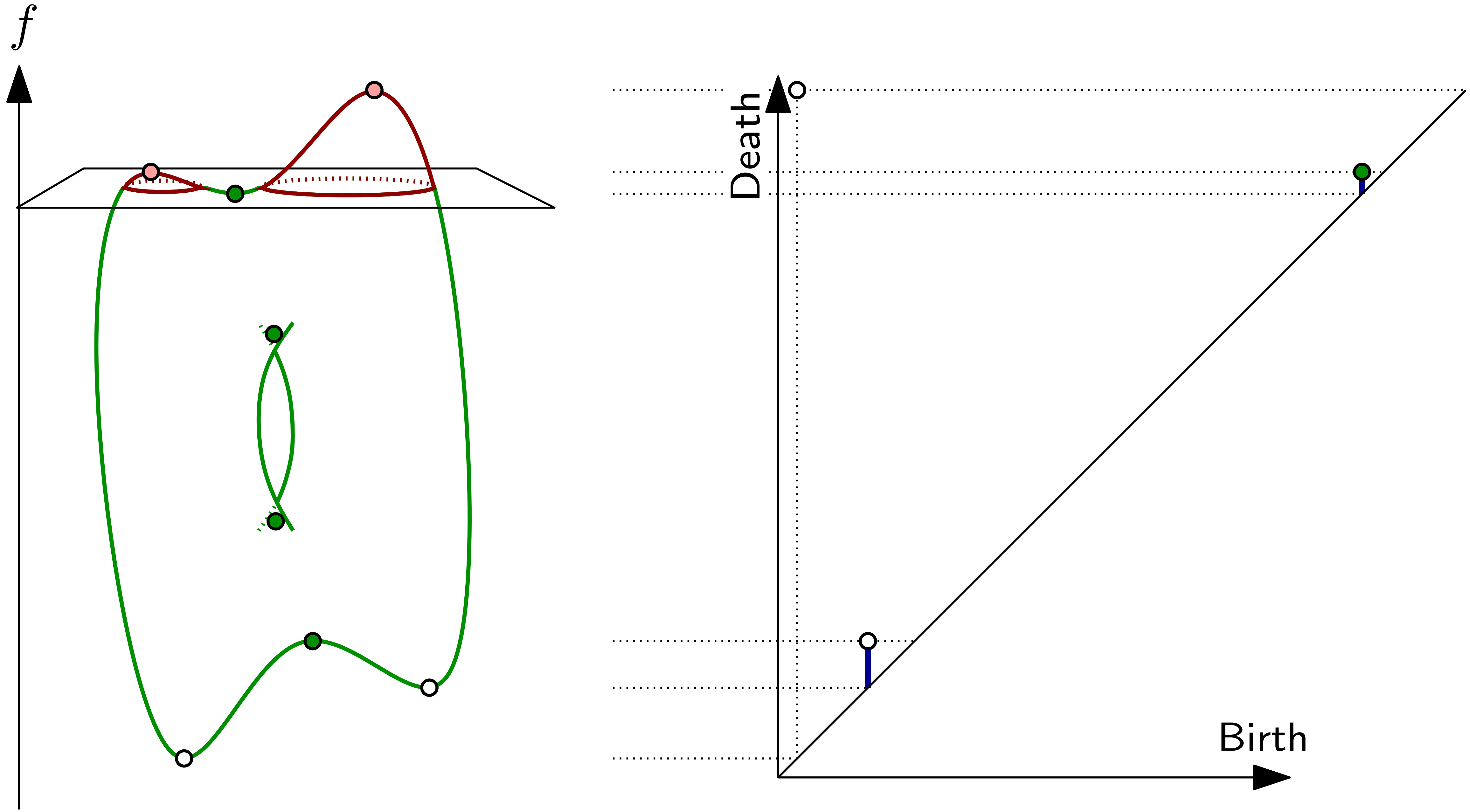




$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]



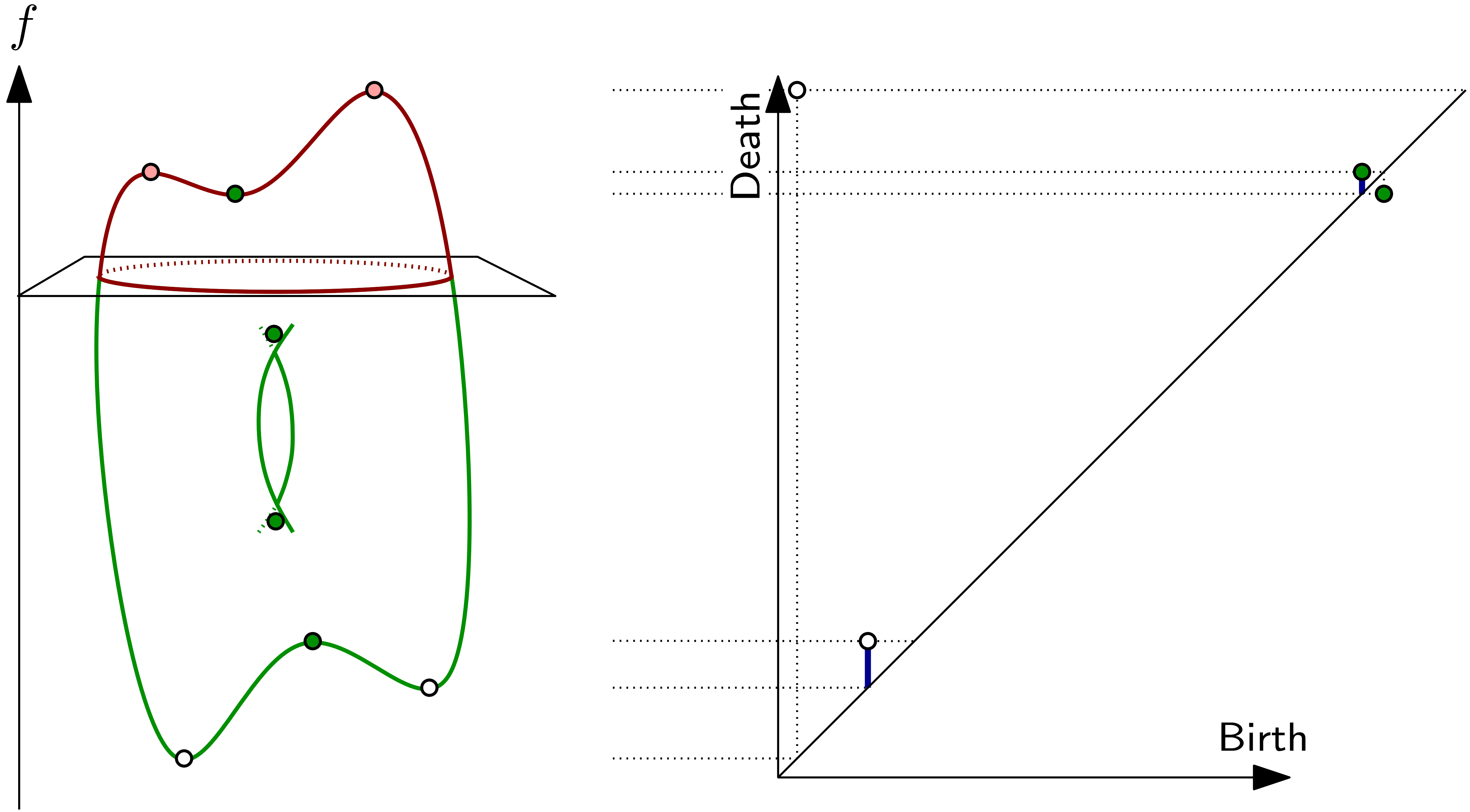
$$\mathsf{H}(\mathbb{X}_1) \rightarrow \mathsf{H}(\mathbb{X}_2) \rightarrow \dots \rightarrow \mathsf{H}(\mathbb{X})$$

$$\downarrow$$

$$\mathsf{H}(\mathbb{X}, \mathbb{X}^1) \leftarrow \dots \leftarrow \mathsf{H}(\mathbb{X}, \mathbb{X}^n) \leftarrow \mathsf{H}(\mathbb{X}, \emptyset)$$

Extended Persistence

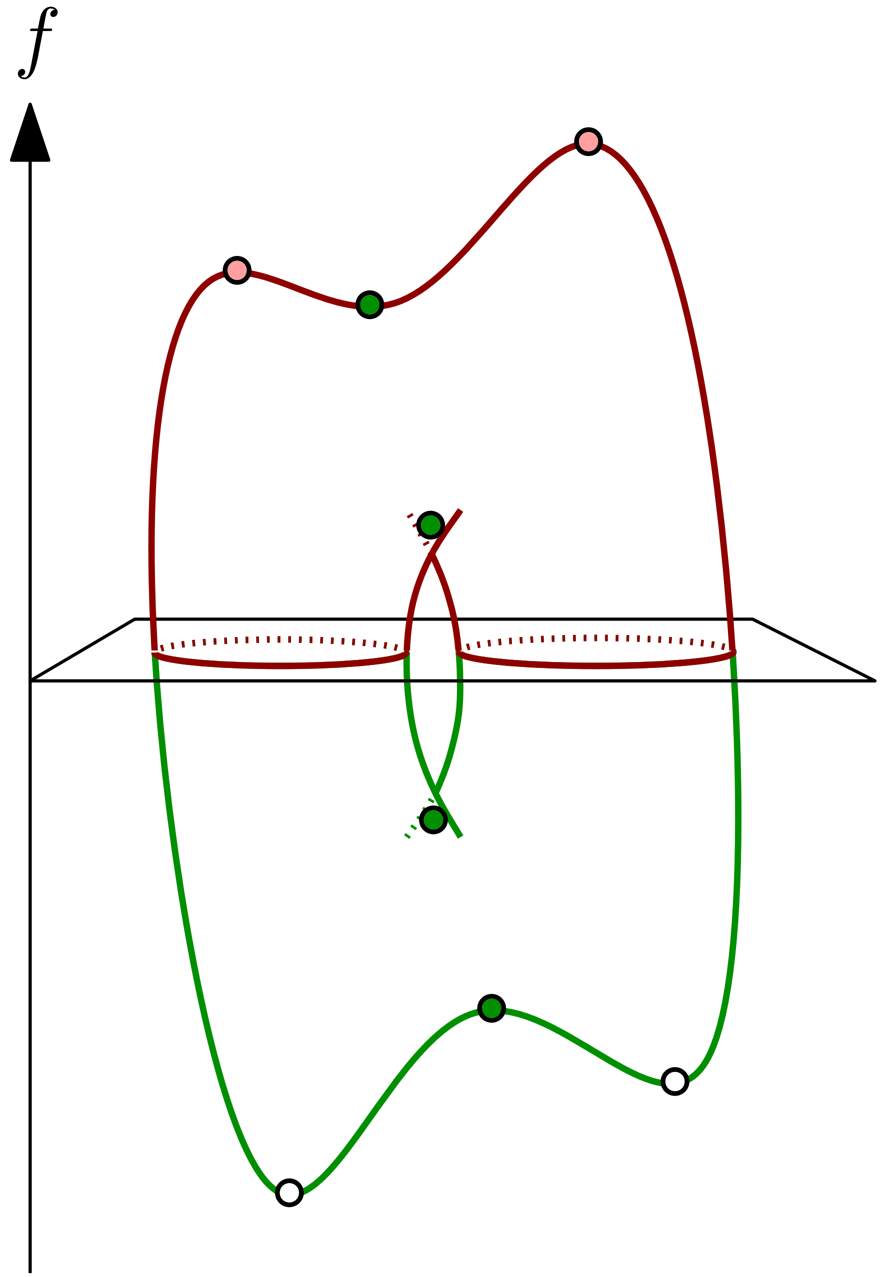
[Cohen-Steiner, Edelsbrunner, Harer '05]

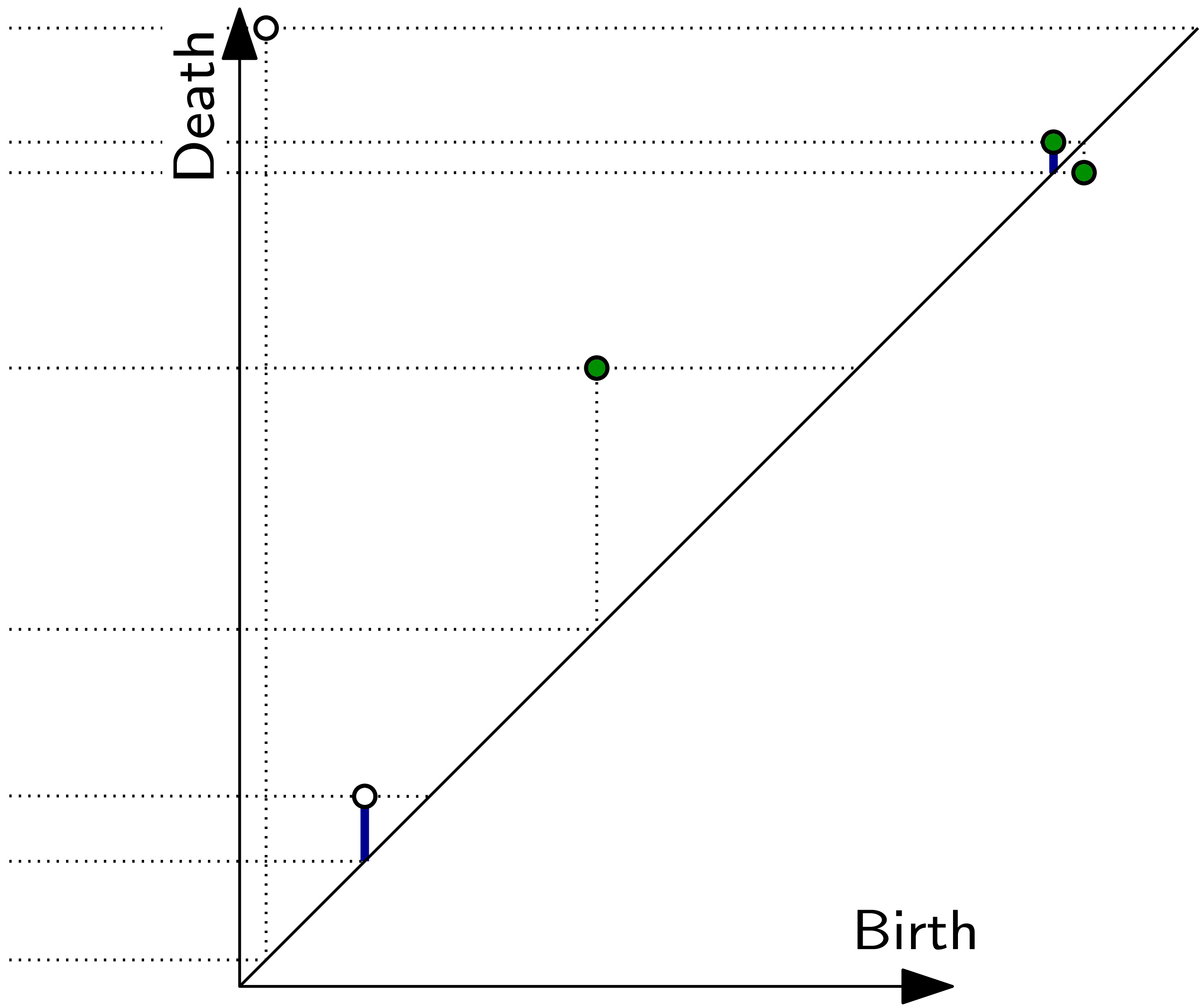


$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
& & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]

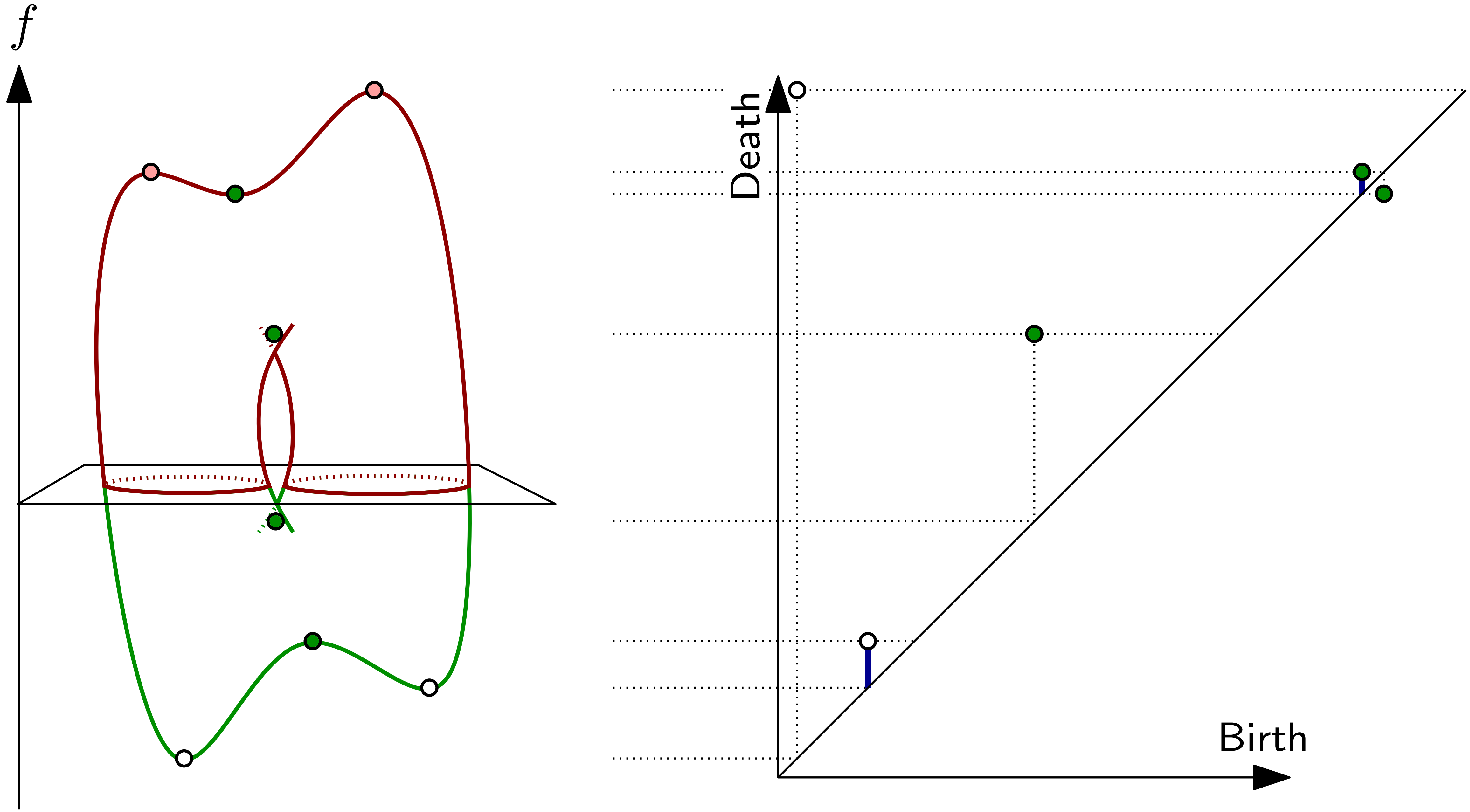




$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

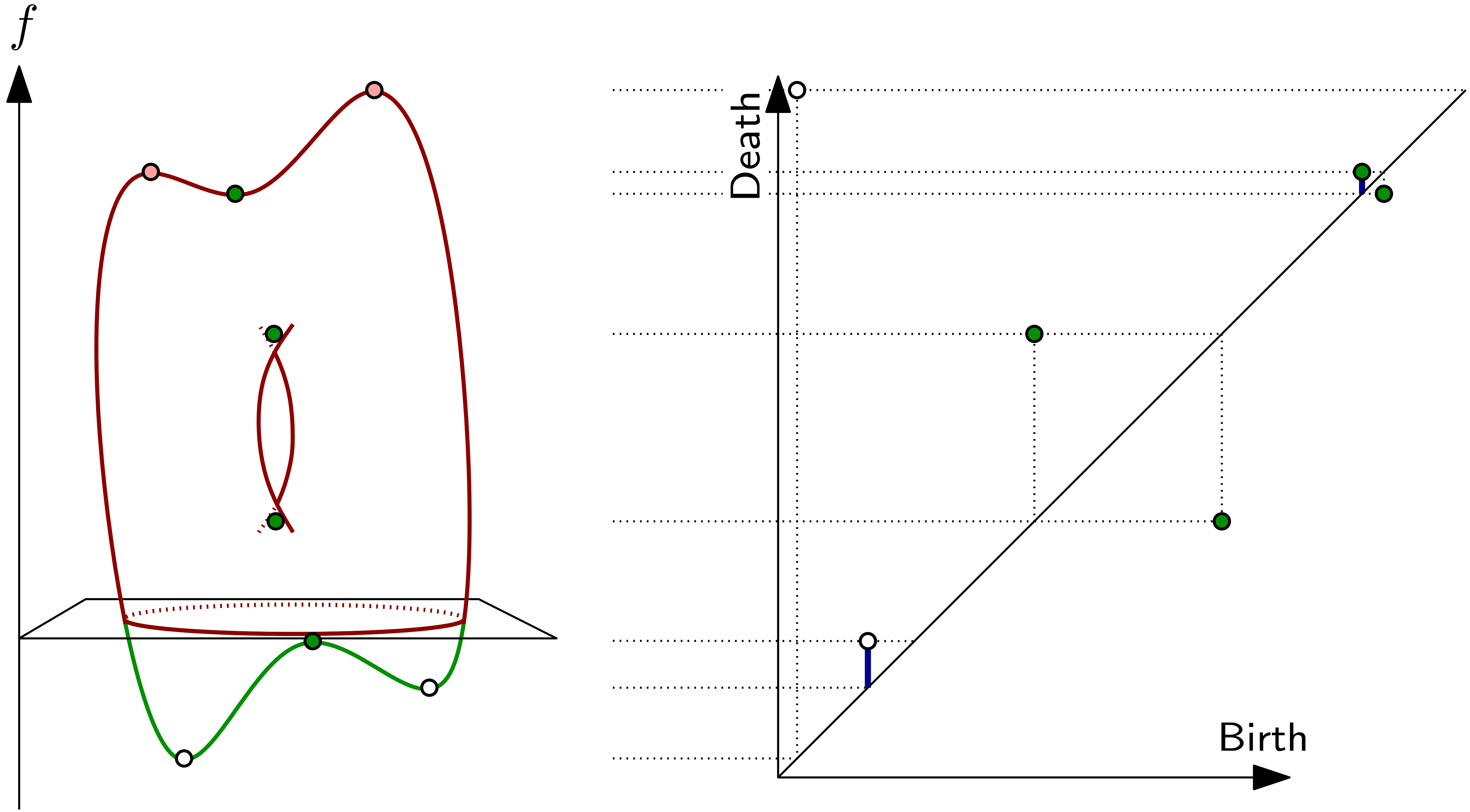
[Cohen-Steiner, Edelsbrunner, Harer '05]



$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

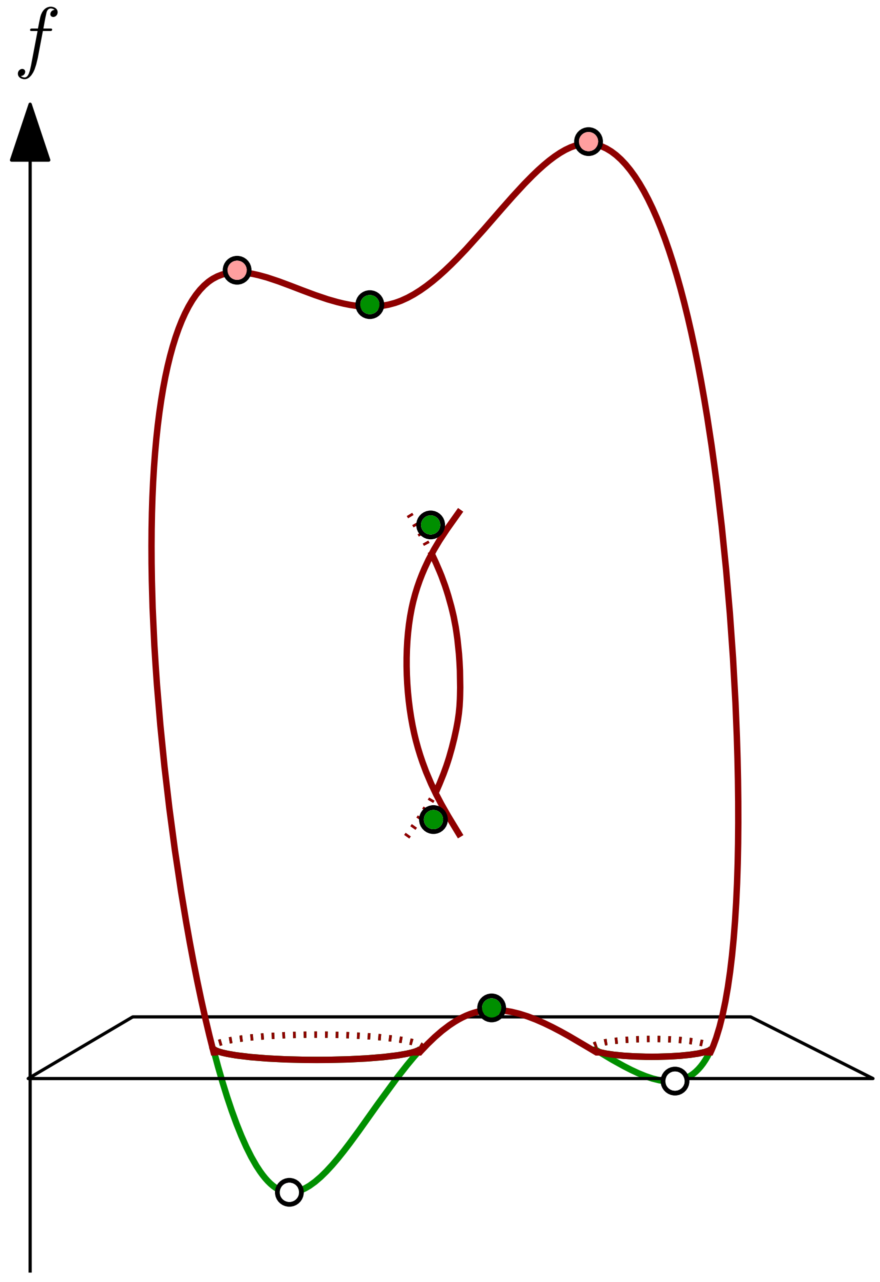
[Cohen-Steiner, Edelsbrunner, Harer '05]

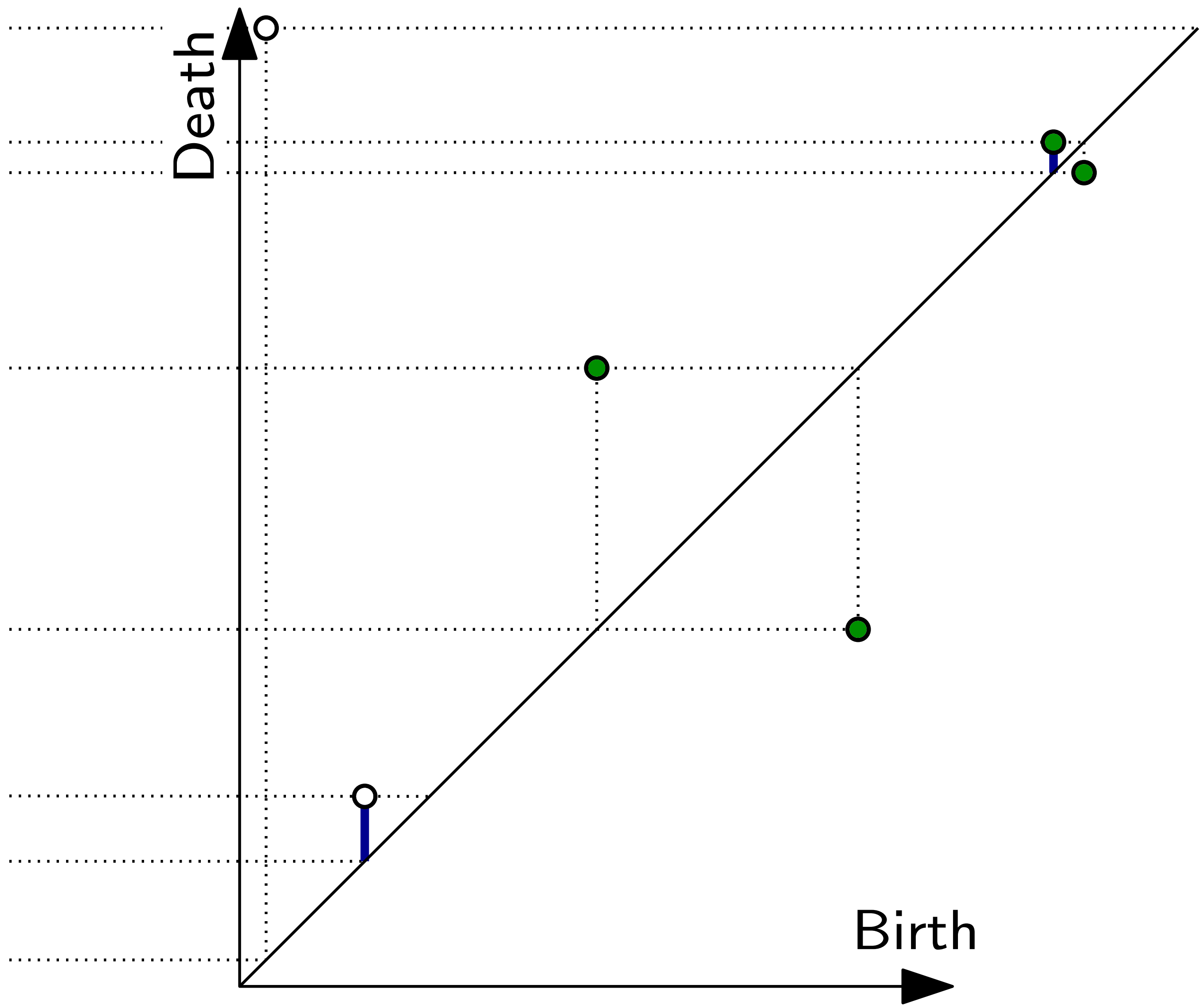


$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]

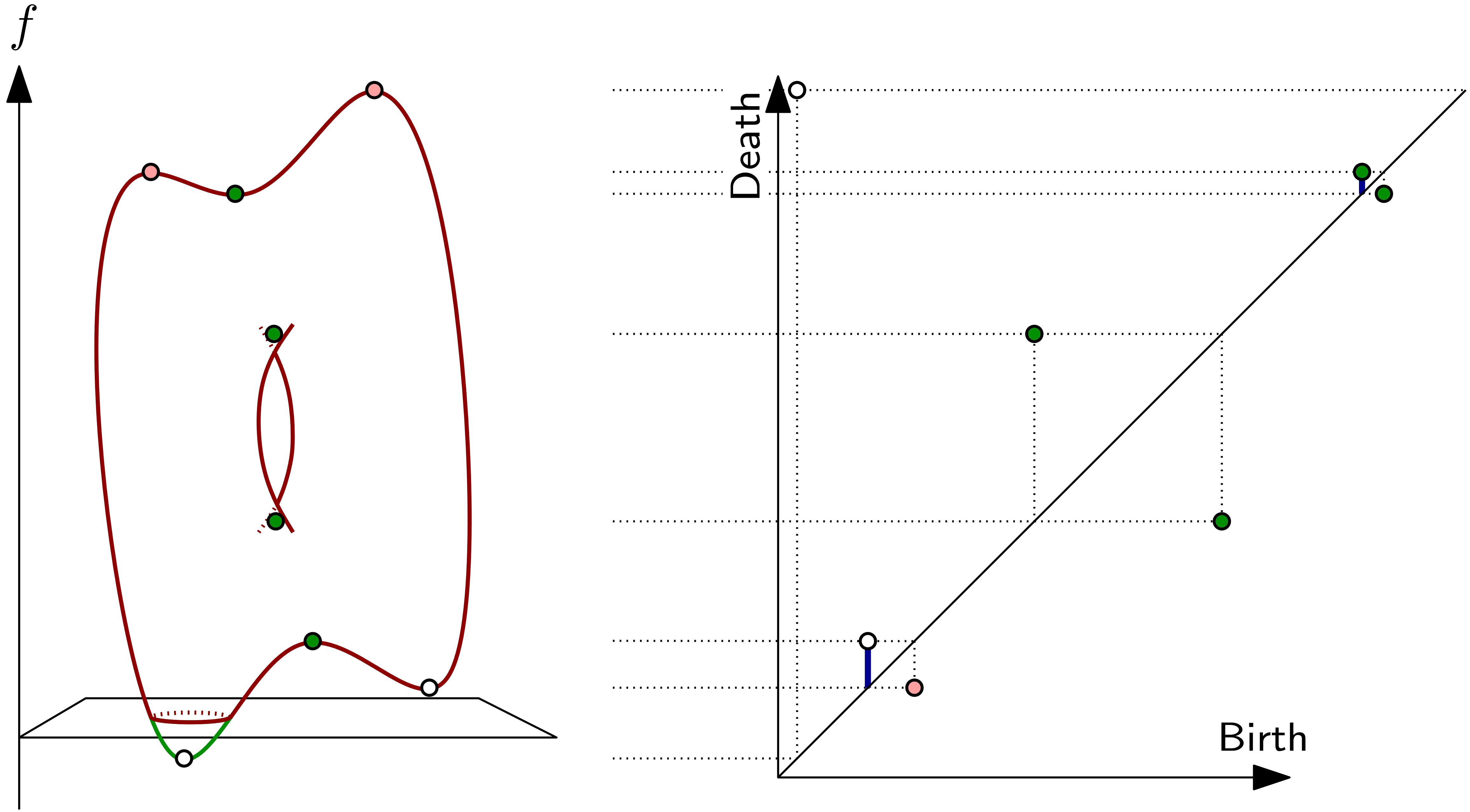




$$\begin{array}{ccccccc} \mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\ & & & & & & \downarrow \\ \mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset) \end{array}$$

Extended Persistence

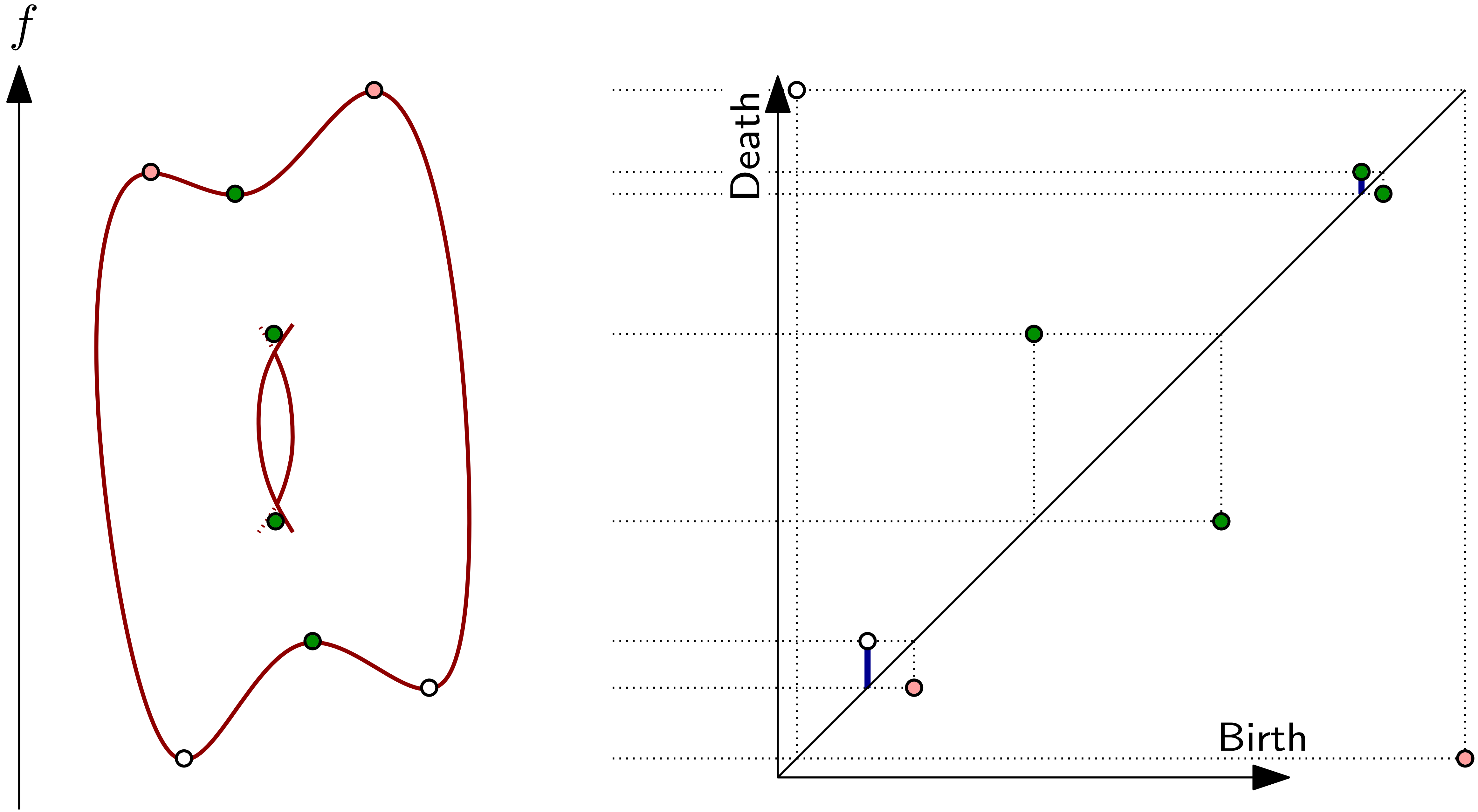
[Cohen-Steiner, Edelsbrunner, Harer '05]



$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
& & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

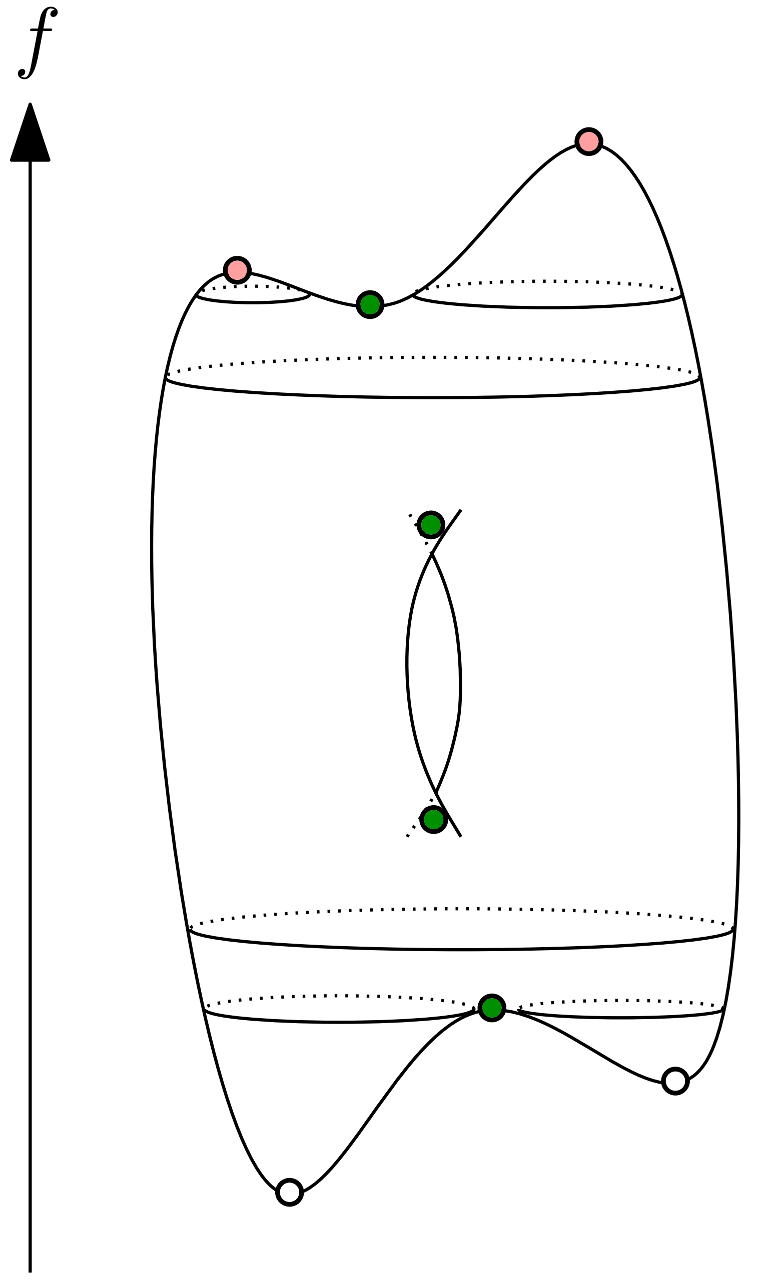
[Cohen-Steiner, Edelsbrunner, Harer '05]

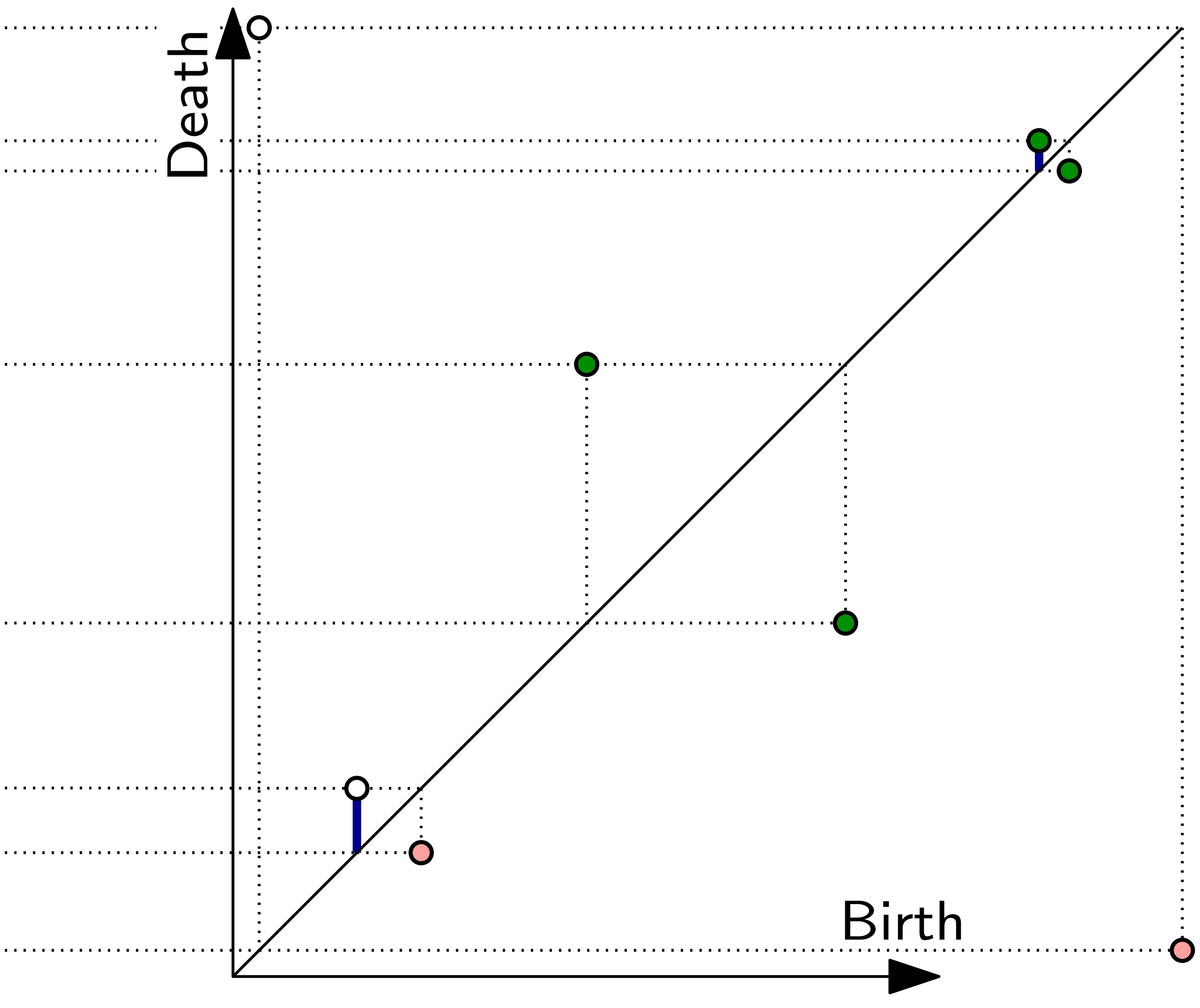


$$
\begin{array}{ccccccc}
\mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\
 & & & & & & \downarrow \\
\mathsf{H}(\mathbb{X}, \mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X}, \mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X}, \emptyset)
\end{array}
$$

Extended Persistence

[Cohen-Steiner, Edelsbrunner, Harer '05]

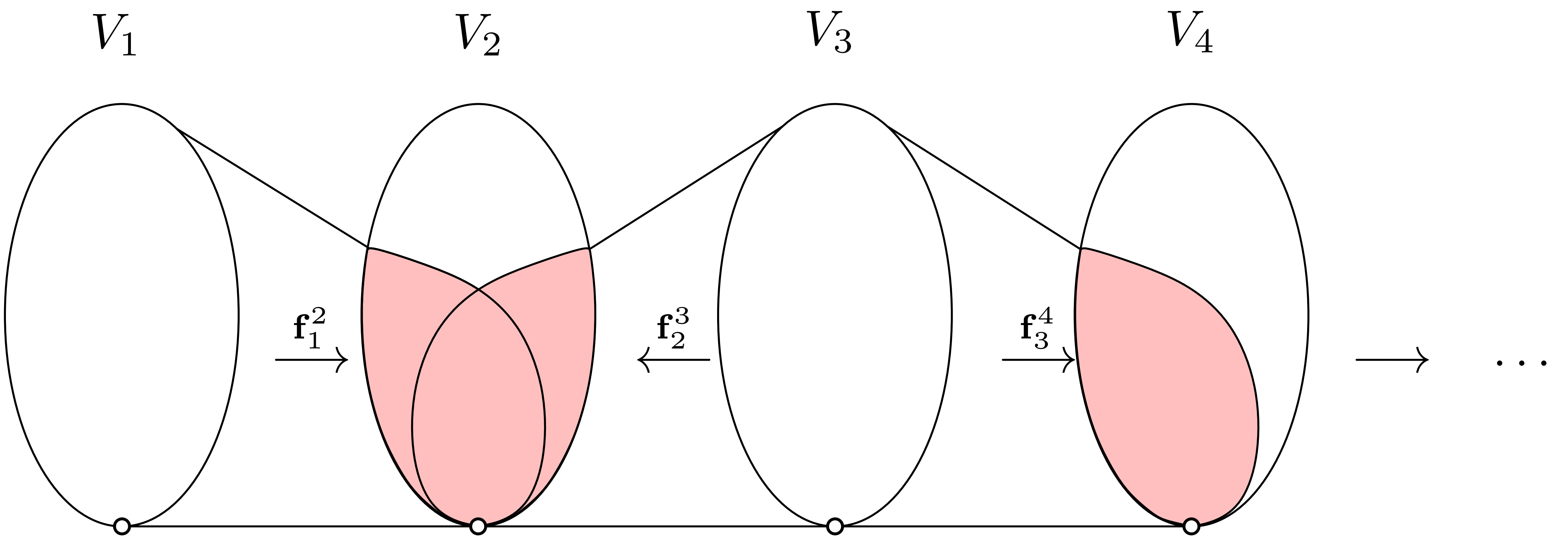




$$\begin{array}{ccccccc} \mathsf{H}(\mathbb{X}_1) & \to & \mathsf{H}(\mathbb{X}_2) & \to & \dots & \to & \mathsf{H}(\mathbb{X}) \\ & & & & & & \downarrow \\ \mathsf{H}(\mathbb{X},\mathbb{X}^1) & \leftarrow & \dots & \leftarrow & \mathsf{H}(\mathbb{X},\mathbb{X}^n) & \leftarrow & \mathsf{H}(\mathbb{X},\emptyset) \end{array}$$

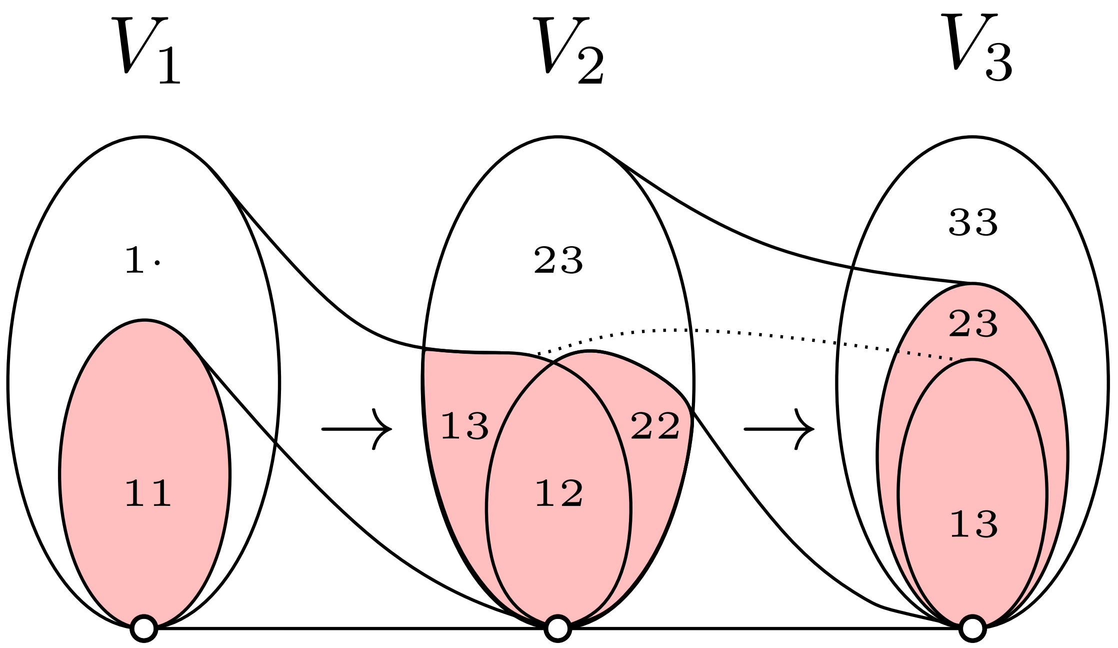
Zigzag Persistence

$$V_1 \leftrightarrow V_2 \leftrightarrow V_3 \leftrightarrow V_4 \leftrightarrow \ldots$$



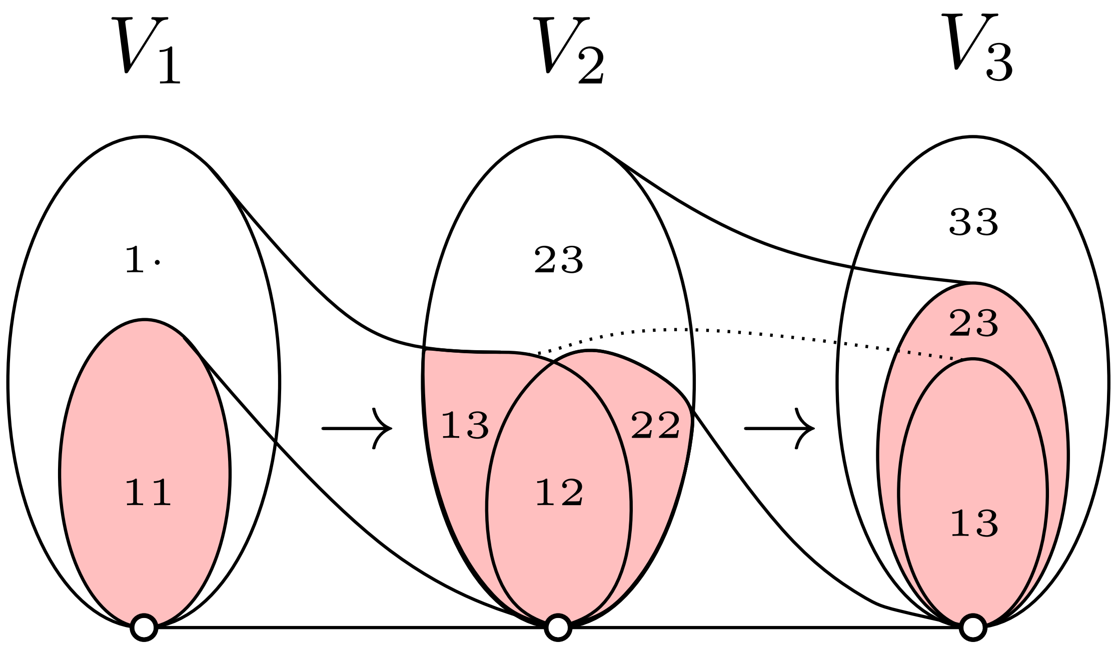
Zigzag Combinations

$$V_1 \to V_2 \to V_3$$

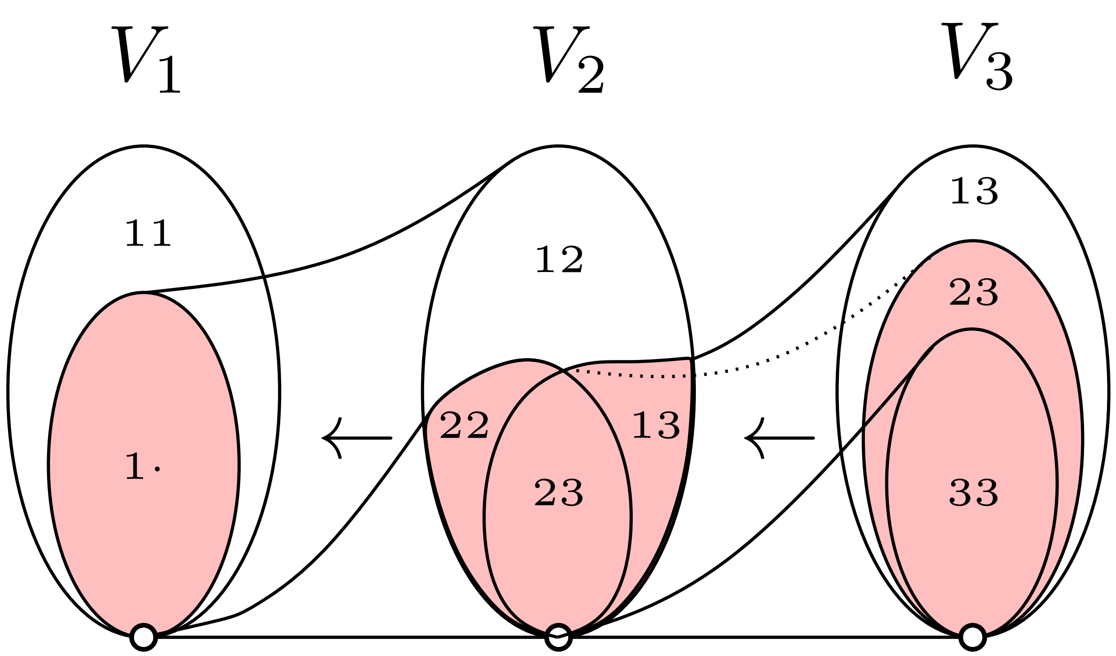


Zigzag Combinations

$$V_1 \rightarrow V_2 \rightarrow V_3$$

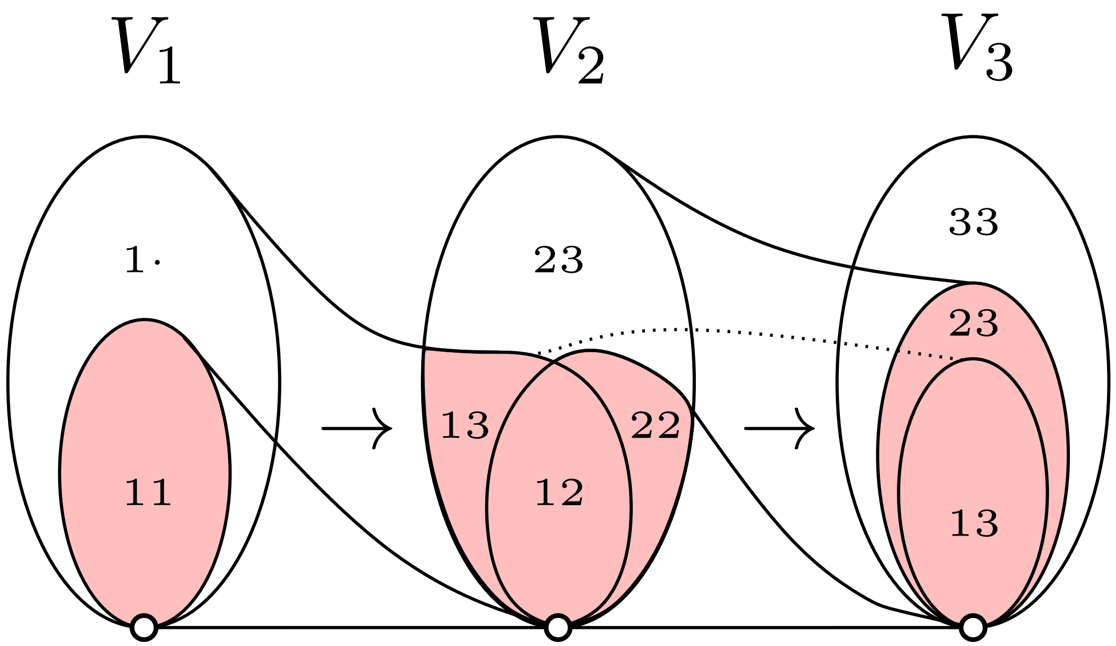


$$V_1 \leftarrow V_2 \leftarrow V_3$$

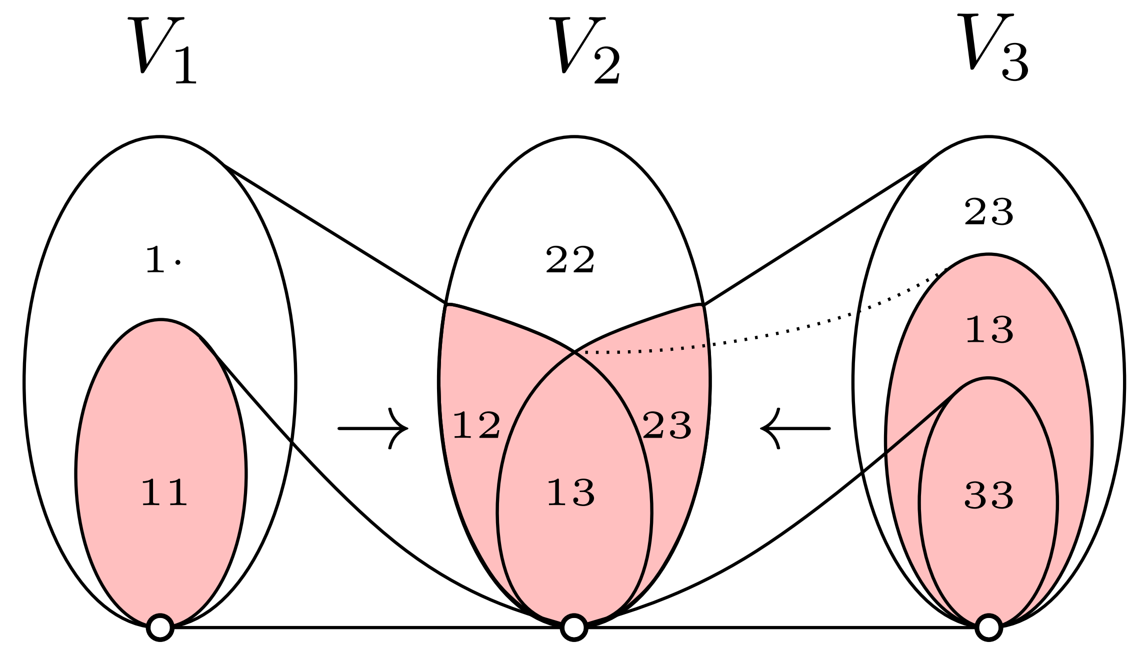


Zigzag Combinations

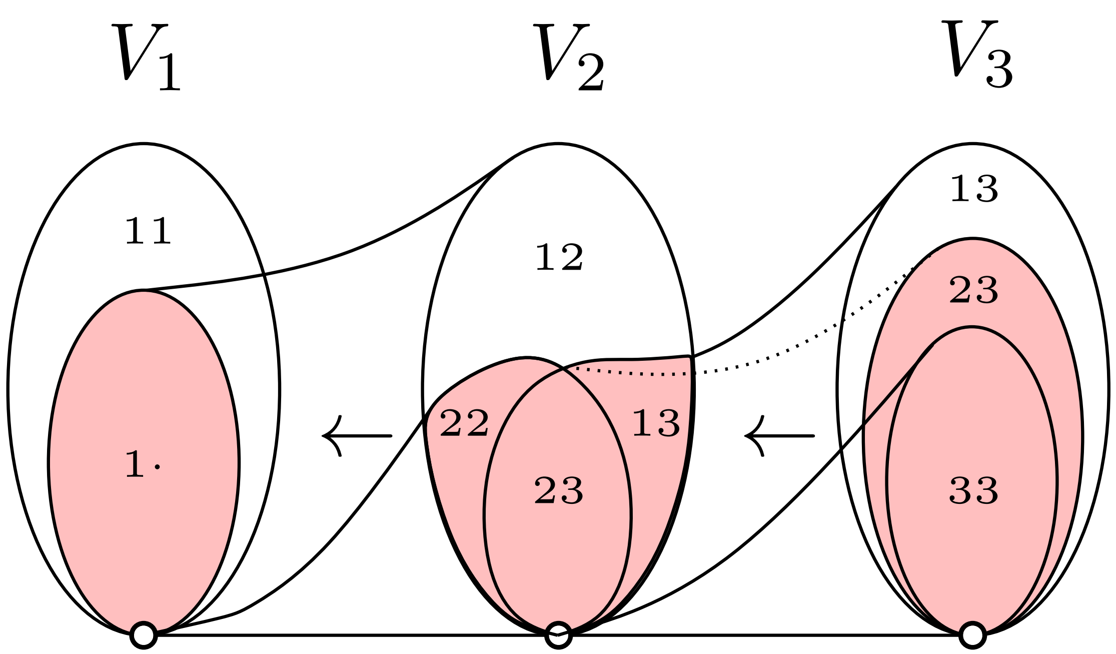
$V_1 \to V_2 \to V_3$



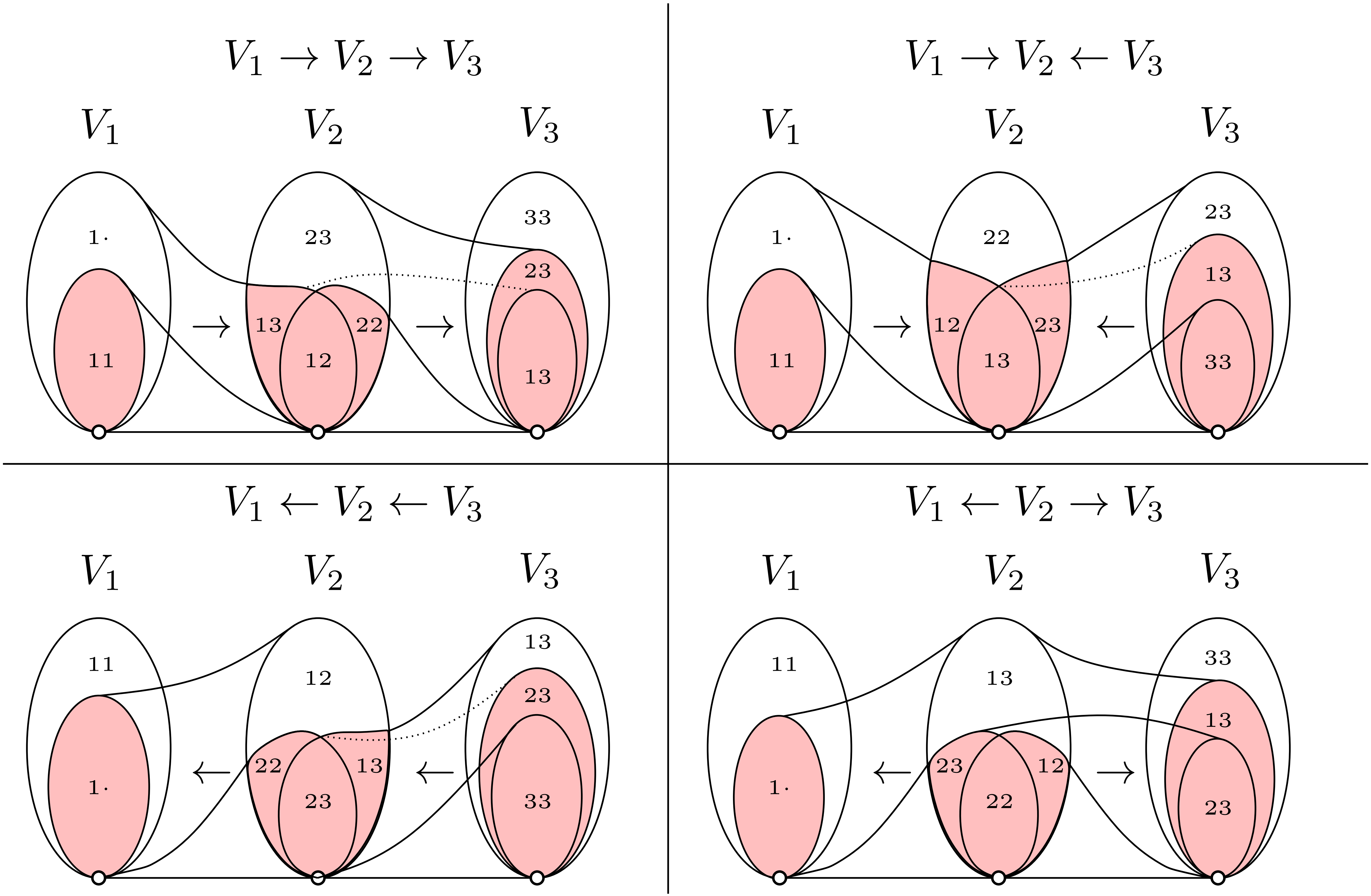
$V_1 \to V_2 \leftarrow V_3$



$V_1 \leftarrow V_2 \leftarrow V_3$



Zigzag Combinations



Zigzag Persistence

$$\mathbb{V}: \quad V_1 \quad \rightarrow \quad V_2 \quad \leftarrow \quad V_3 \quad \rightarrow \quad V_4 \quad \leftarrow \quad V_5 \quad \rightarrow \quad \ldots$$

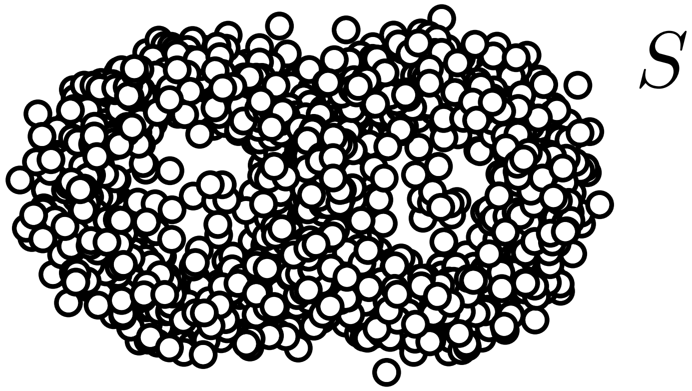
$$\mathbb{I}_{[2,4]}: \quad 0 \quad \rightarrow \quad k \quad \xleftarrow{1} \quad k \quad \xrightarrow{1} \quad k \quad \leftarrow \quad 0 \quad \rightarrow \quad \ldots$$

Theorem: Any zigzag module $\mathbb{V}$ can be written as a sum of interval modules $\mathbb{I}$:

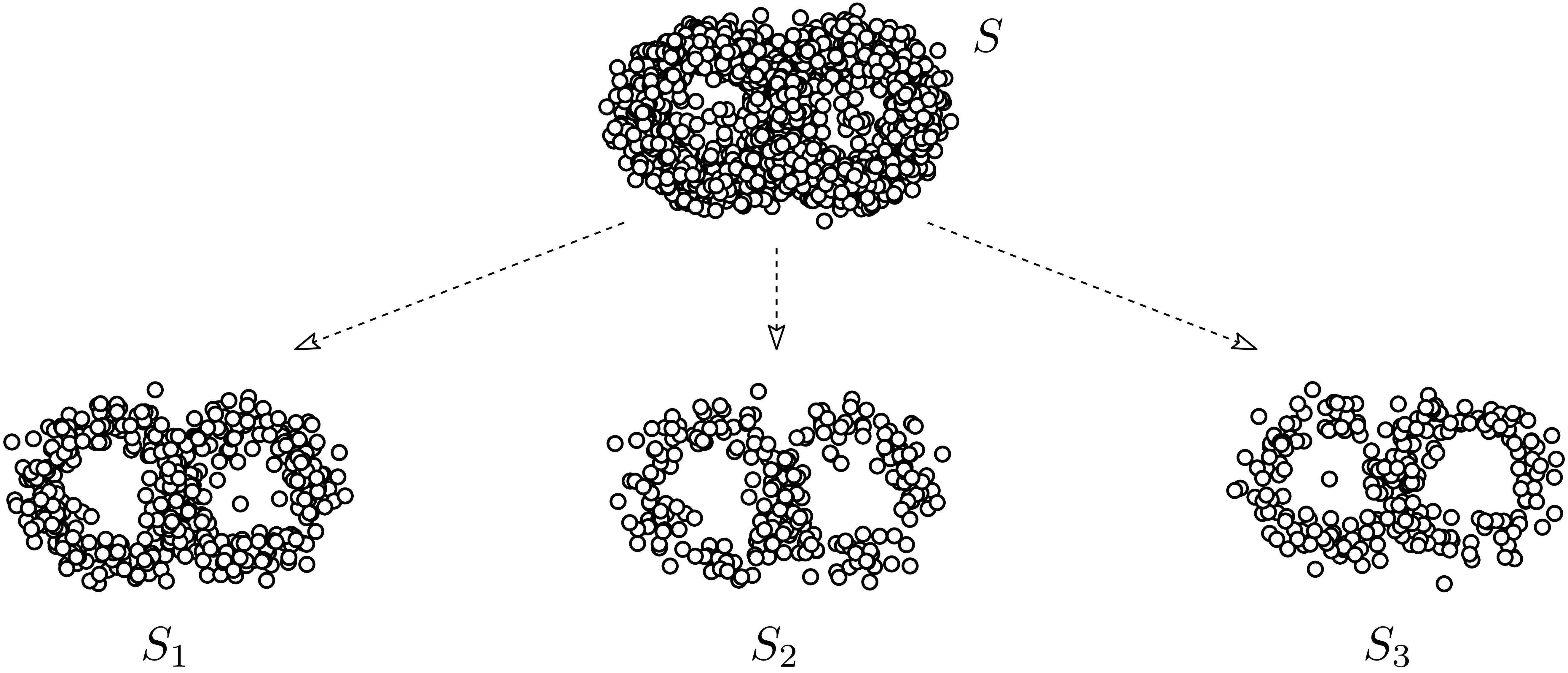
$$\mathbb{V} = \bigoplus_{a \in A} \mathbb{I}_{[b_a, d_a]}$$

[Carlsson, de Silva '10]

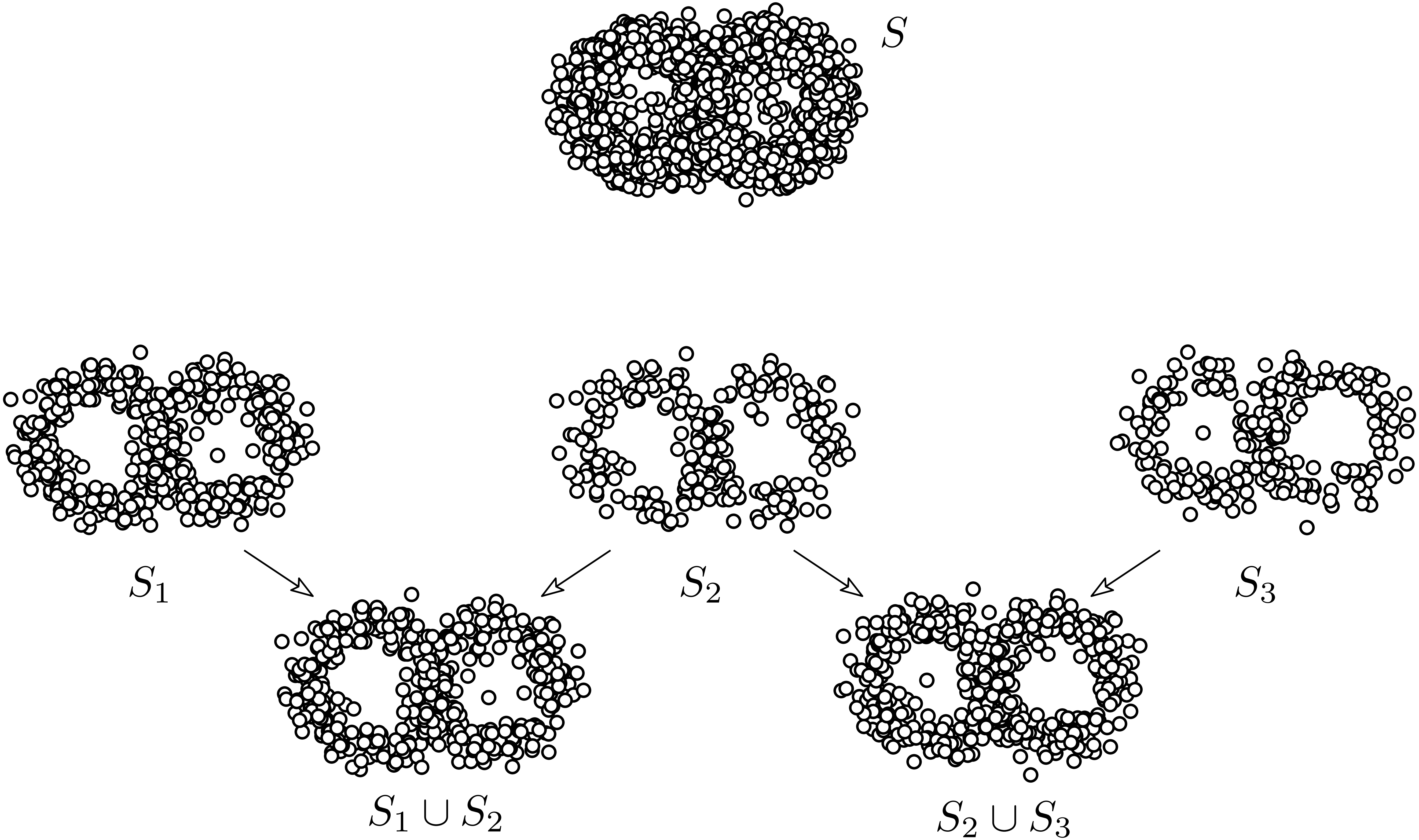
Zigzag Persistence



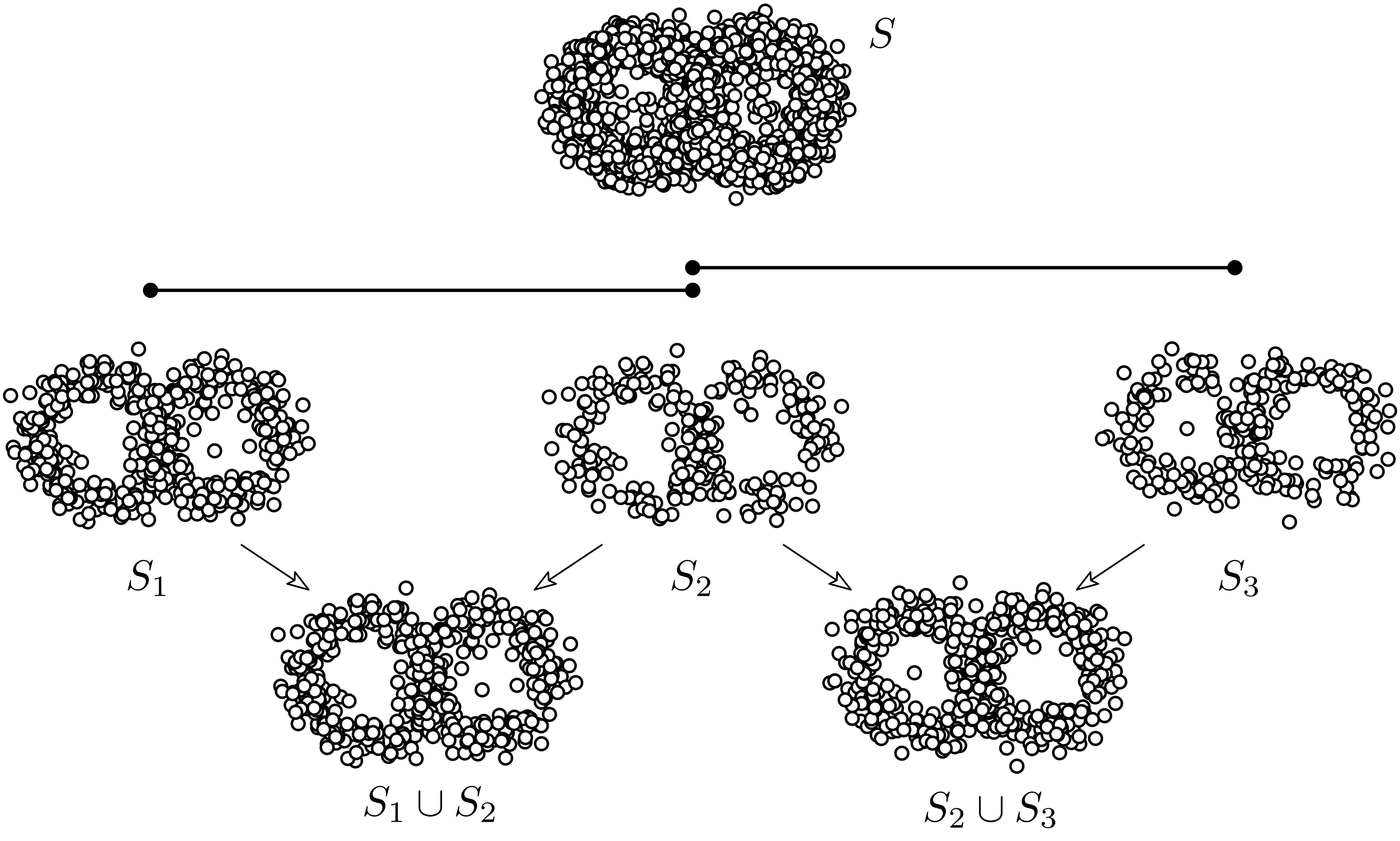
Zigzag Persistence



Zigzag Persistence

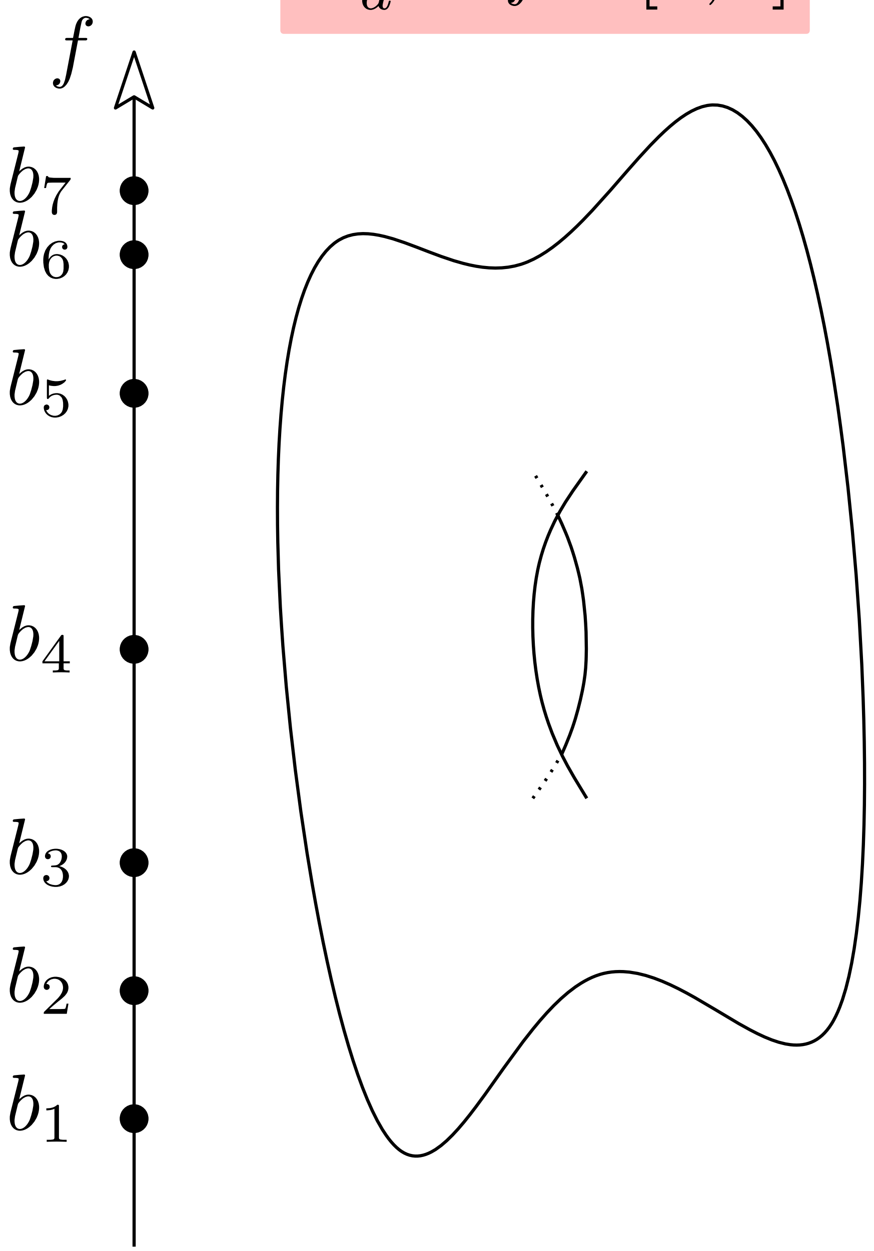


Zigzag Persistence



Levelset Zigzag

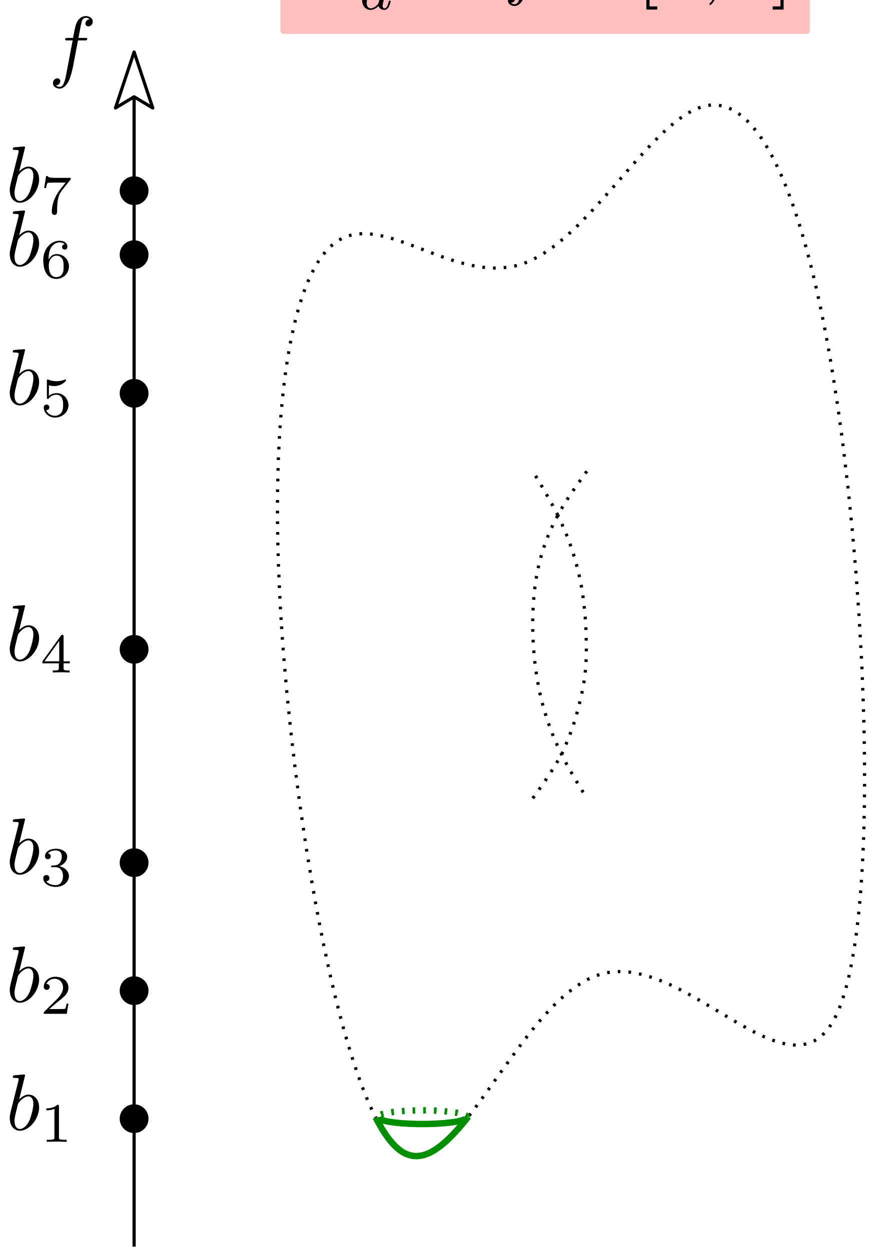
$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$



[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

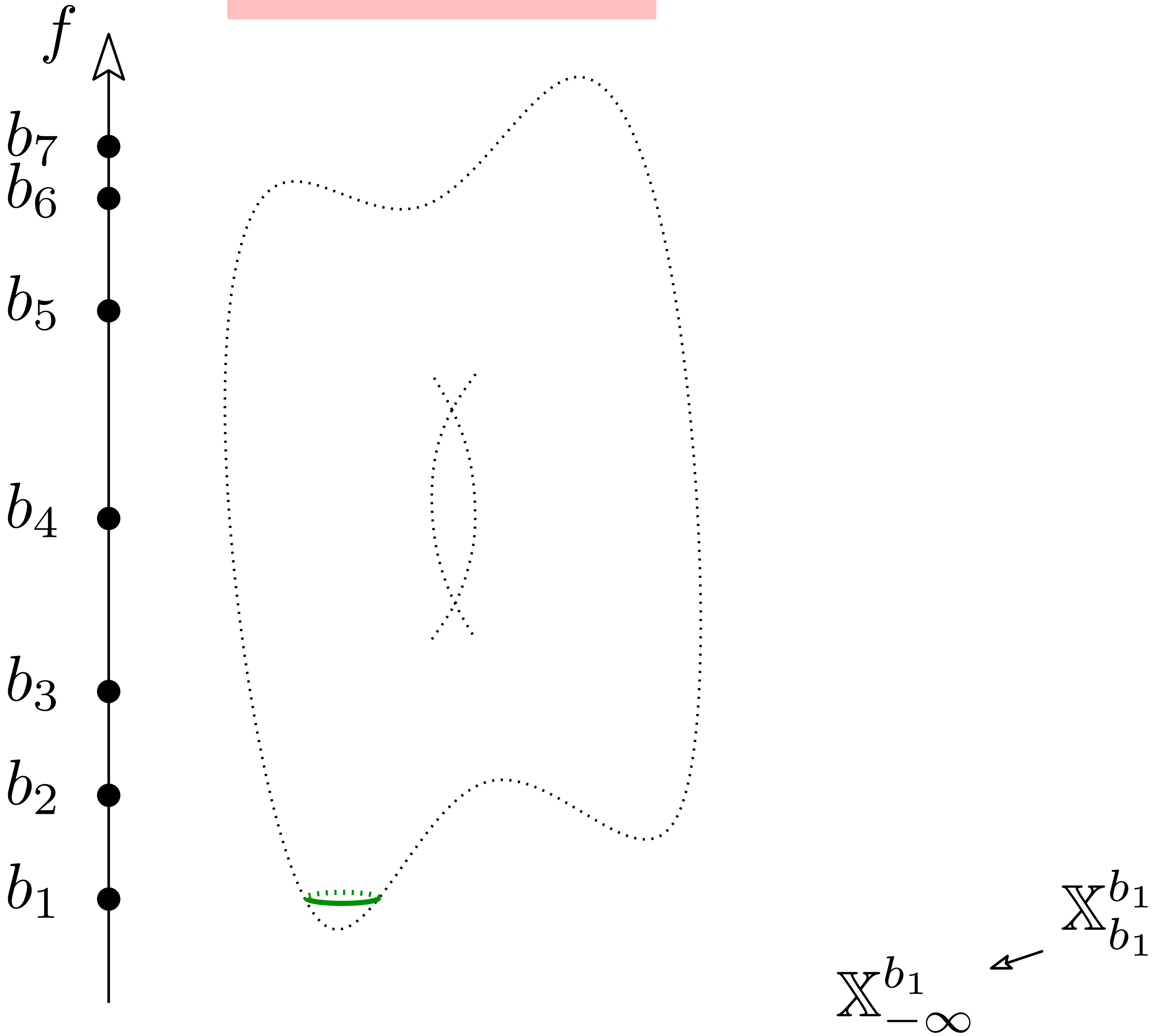


$\mathbb{X}_{-\infty}^{b_1}$

[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

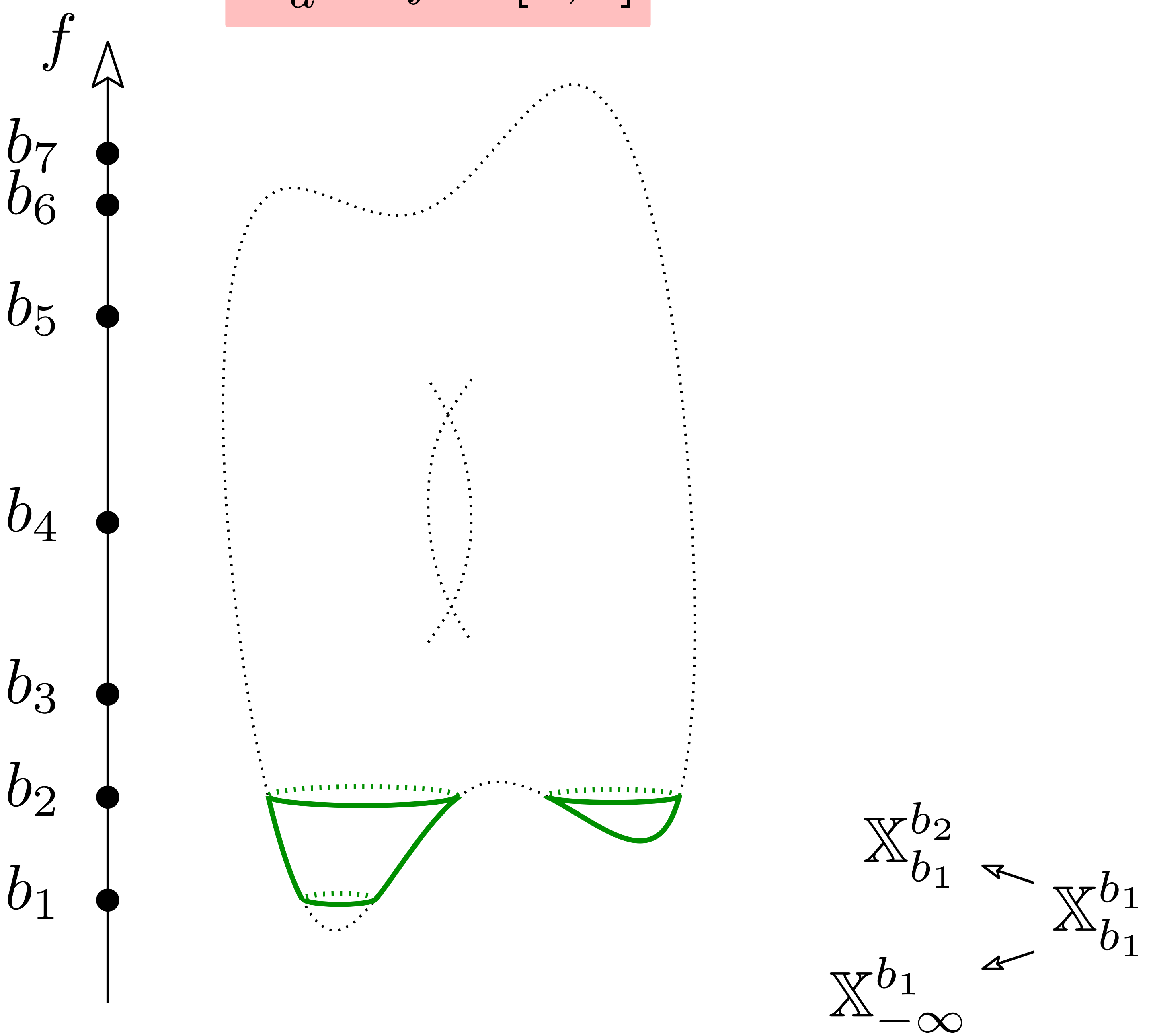


[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$

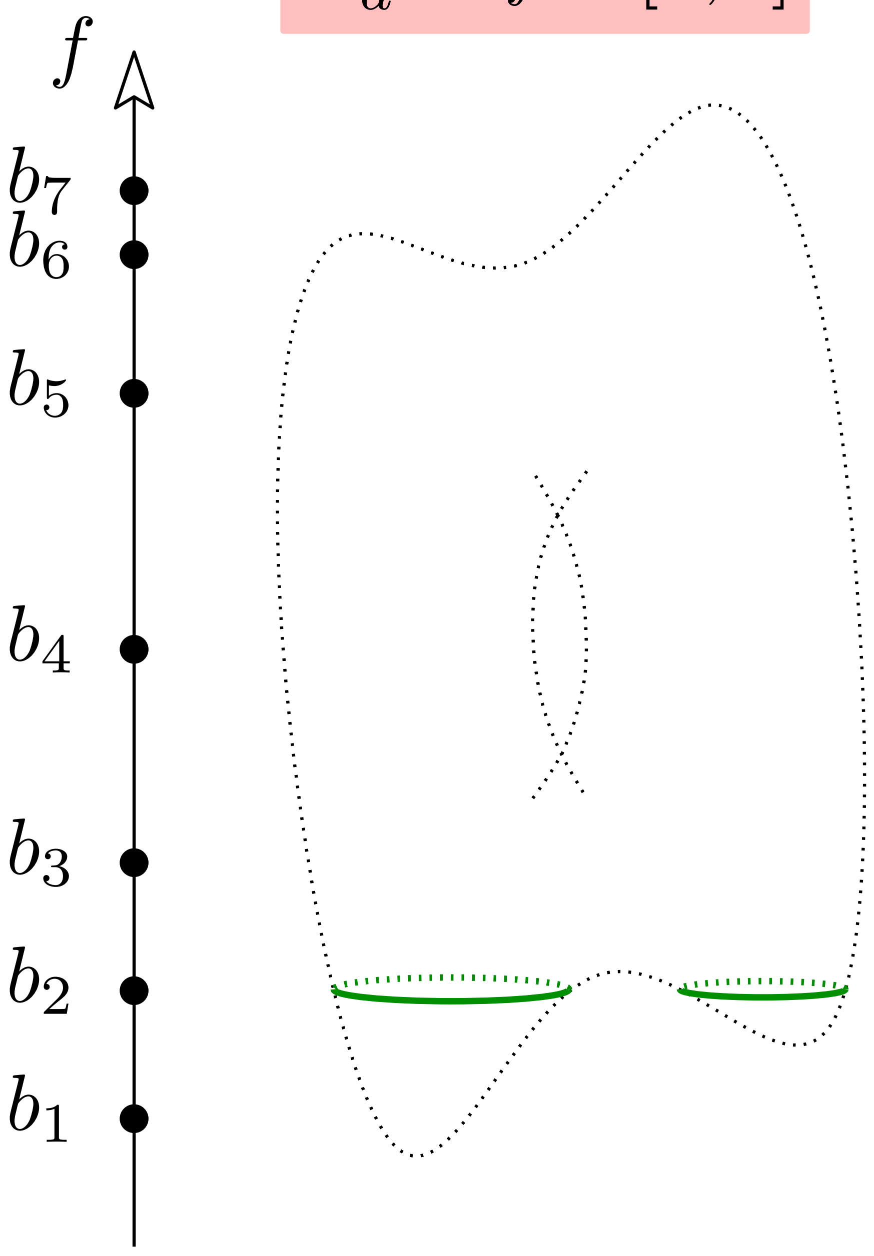
$\mathbb{X}_a^b = f^{-1}[a, b]$



[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

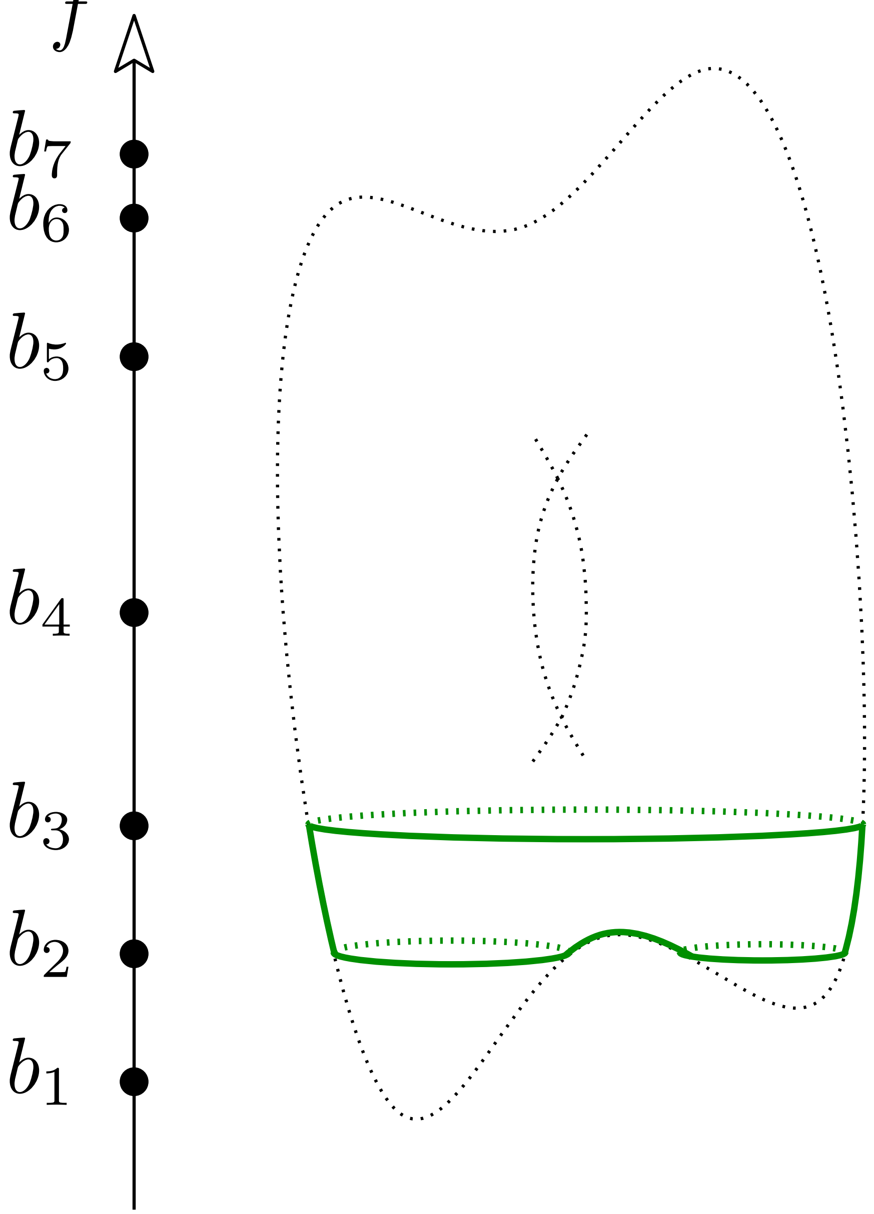


$$\mathbb{X}_{-\infty}^{b_1} \leftarrow \mathbb{X}_{b_1}^{b_1} \rightarrow \mathbb{X}_{b_1}^{b_2} \leftarrow \mathbb{X}_{b_2}^{b_2}$$

[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

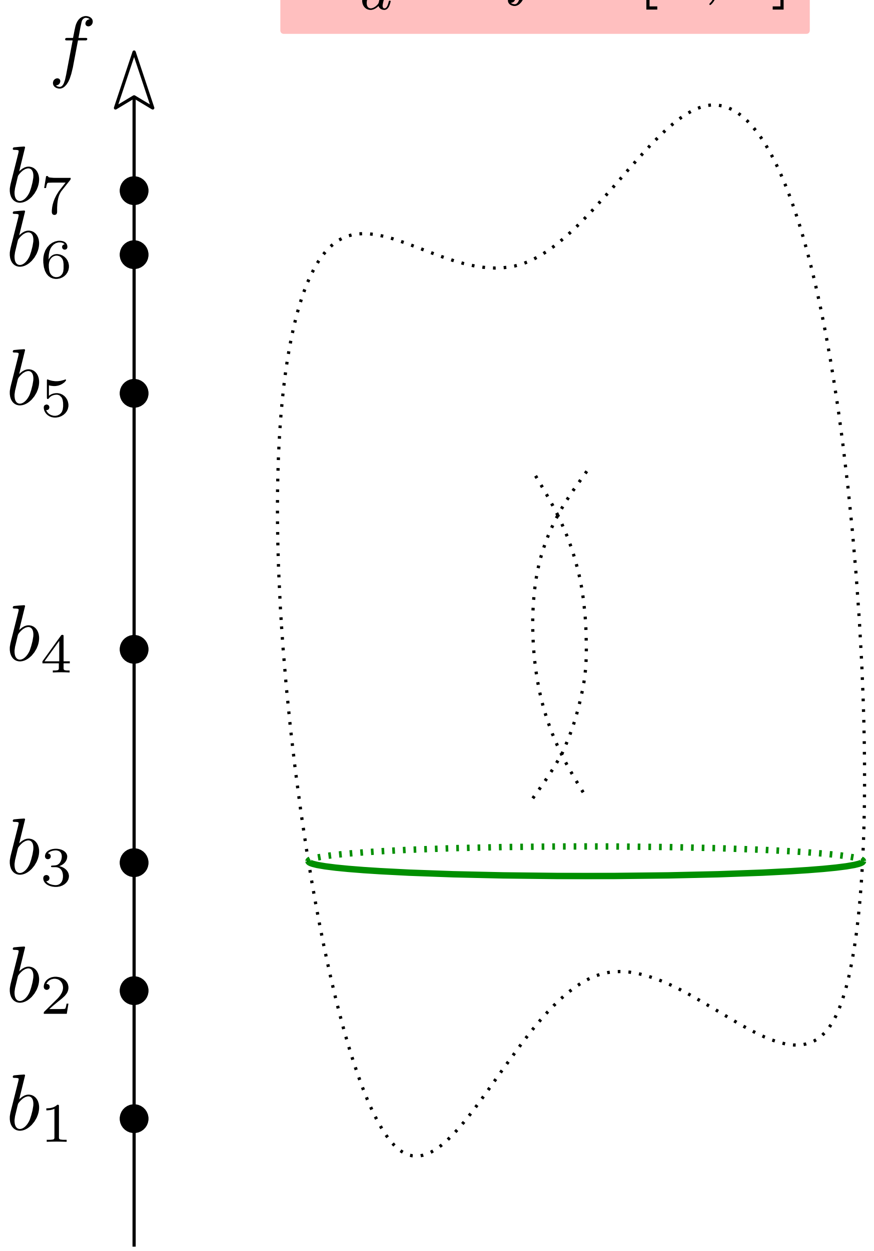


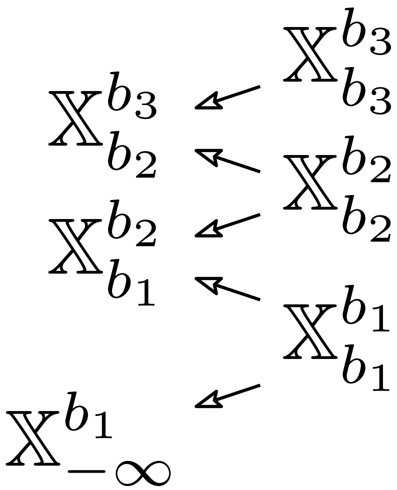
$\mathbb{X}_{b_2}^{b_3} \leftarrow \mathbb{X}_{b_2}^{b_2} \rightarrow \mathbb{X}_{b_1}^{b_2} \leftarrow \mathbb{X}_{b_1}^{b_1} \rightarrow \mathbb{X}_{-\infty}^{b_1}$

[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

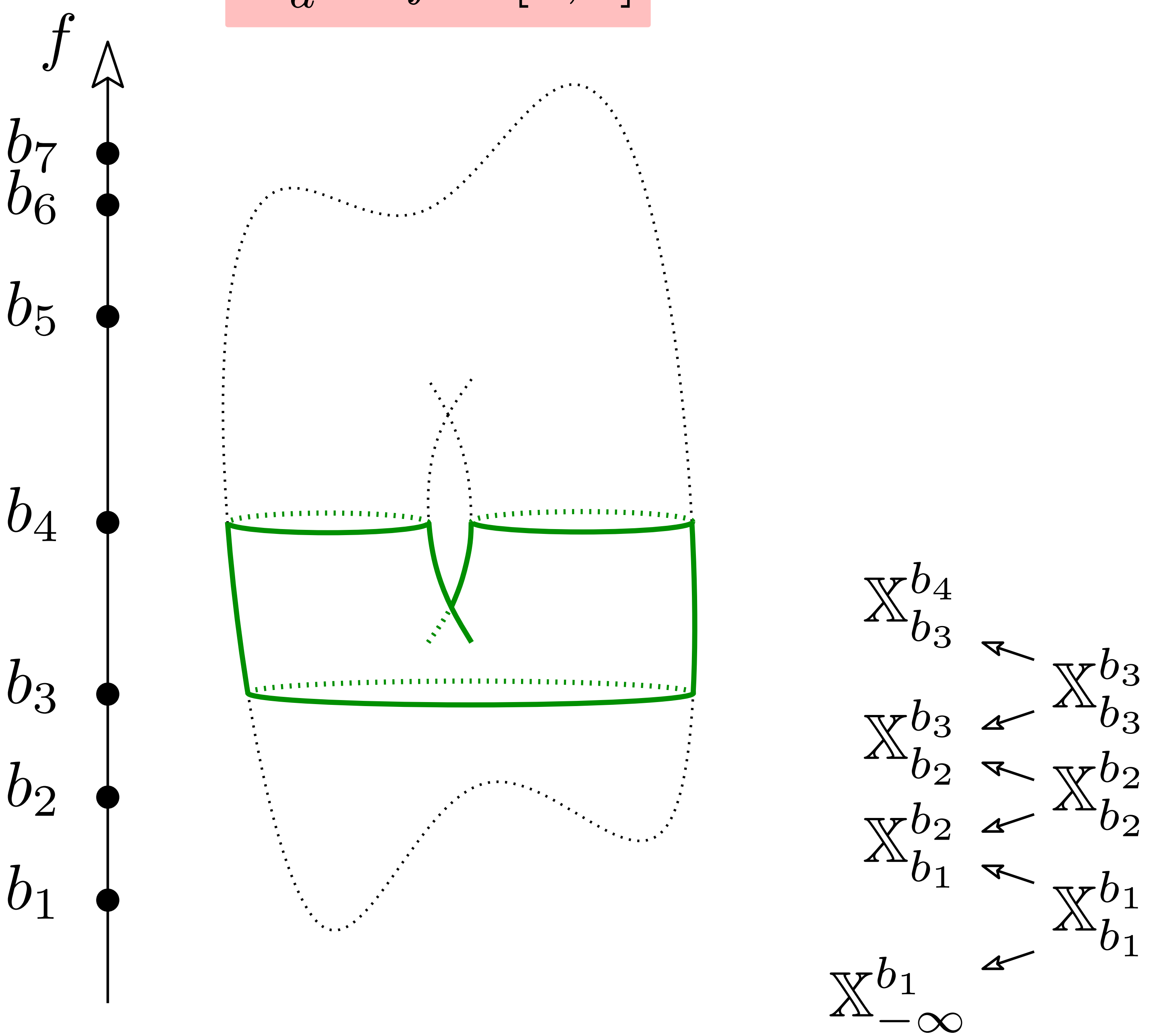




[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

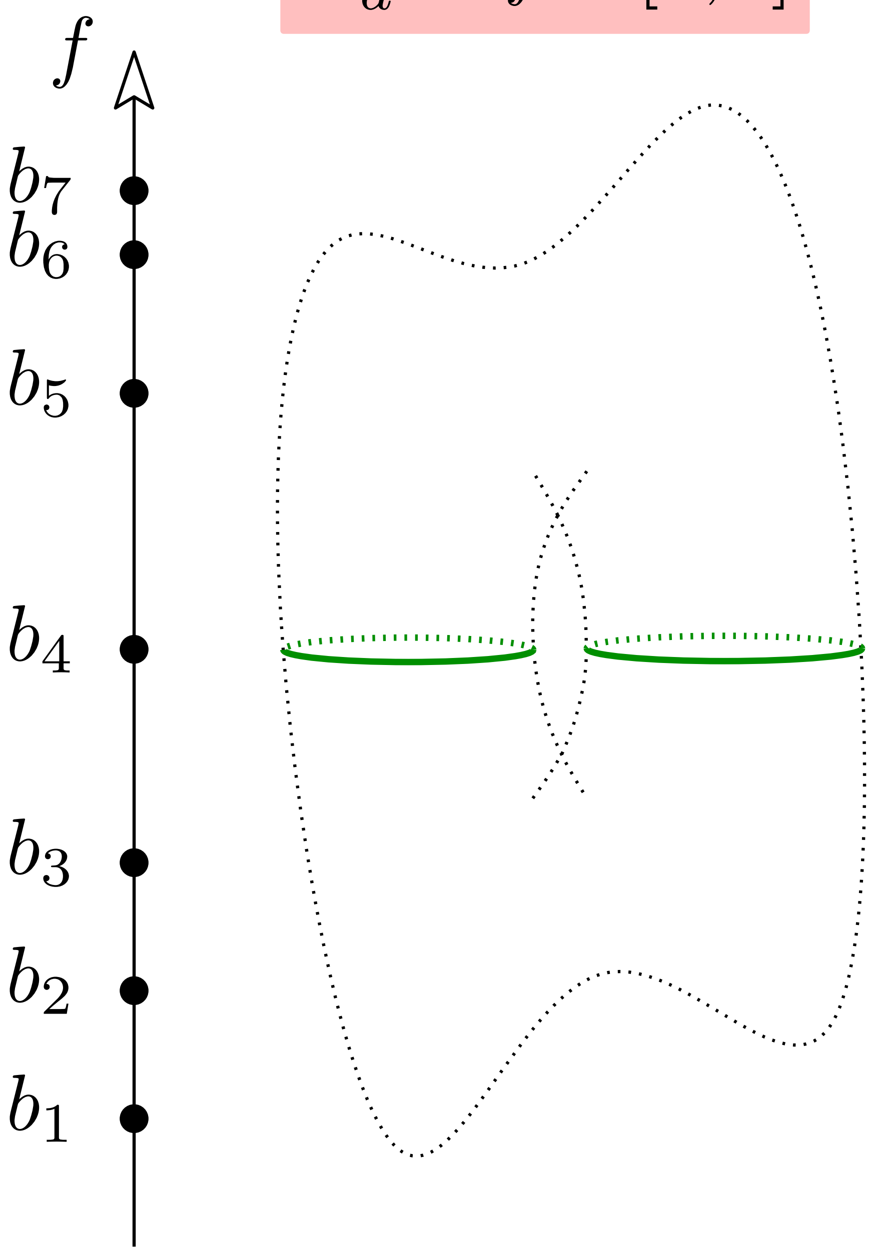
$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

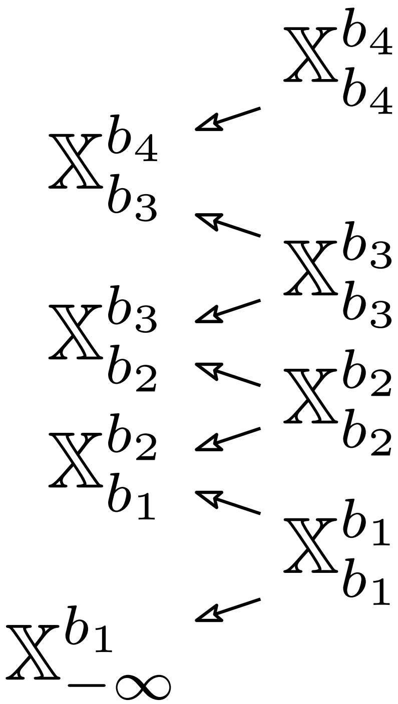


[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

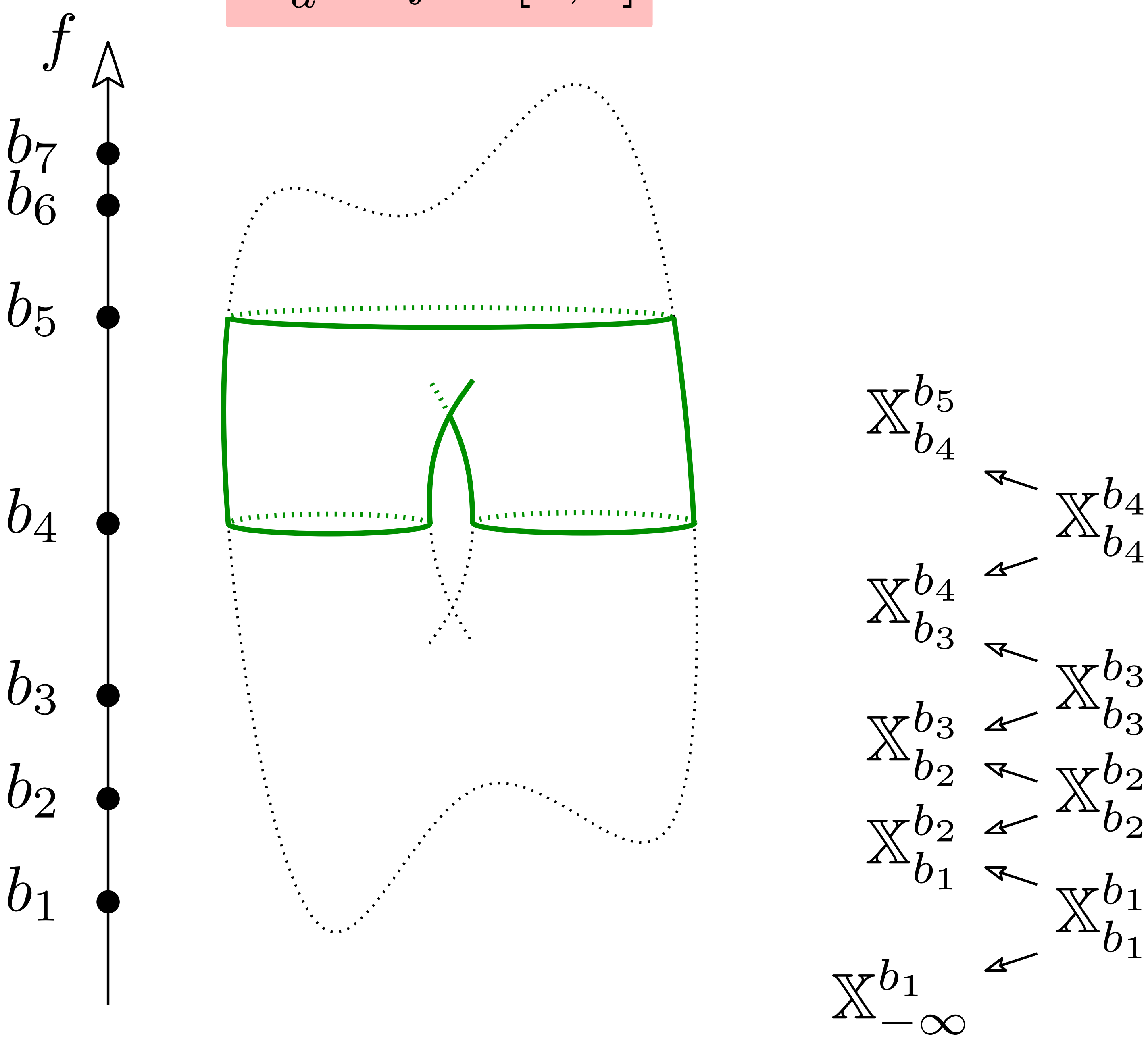




[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

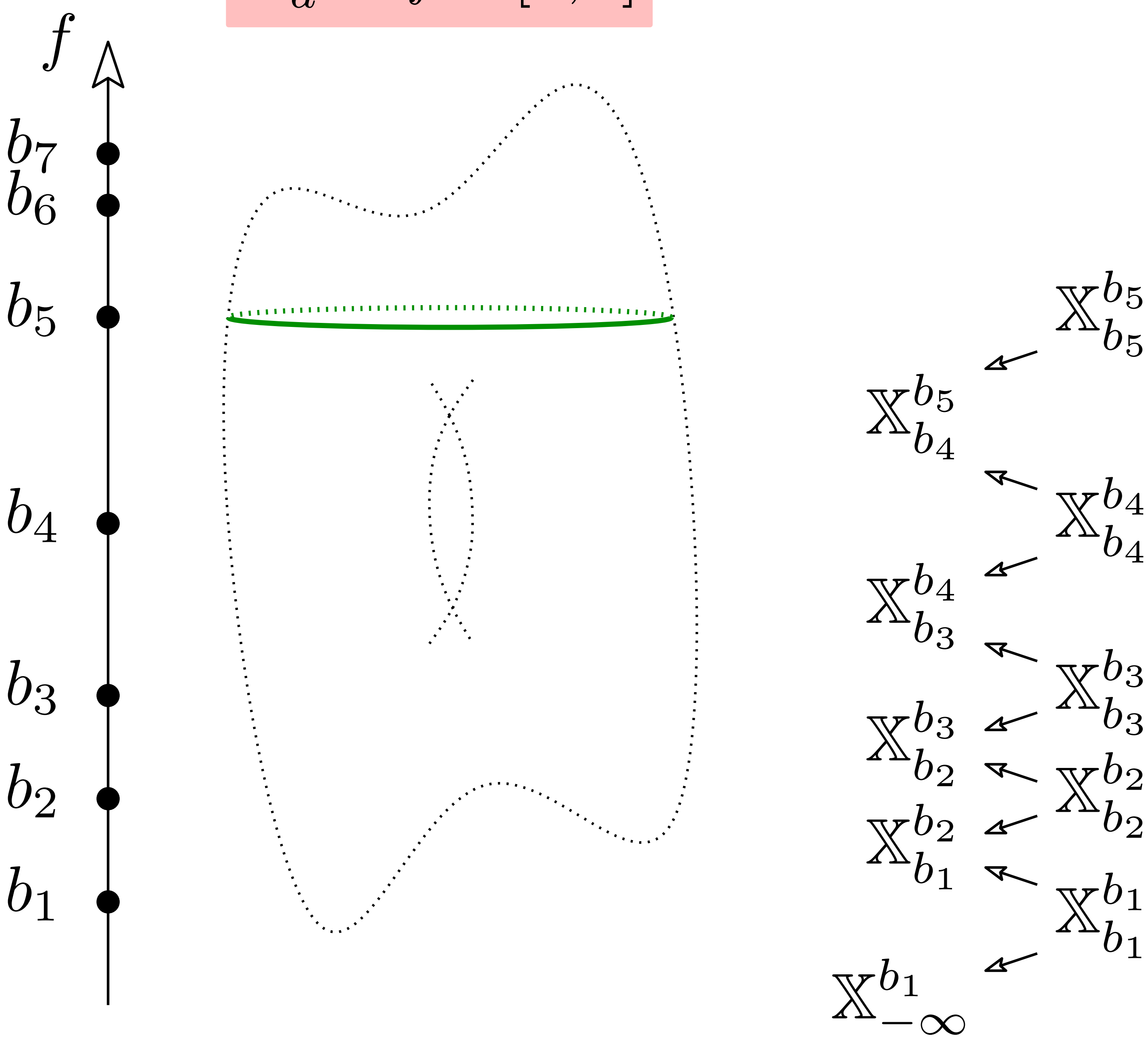
$$f : \mathbb{X} \to \mathbb{R}$$
$$\mathbb{X}_a^b = f^{-1}[a, b]$$



[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

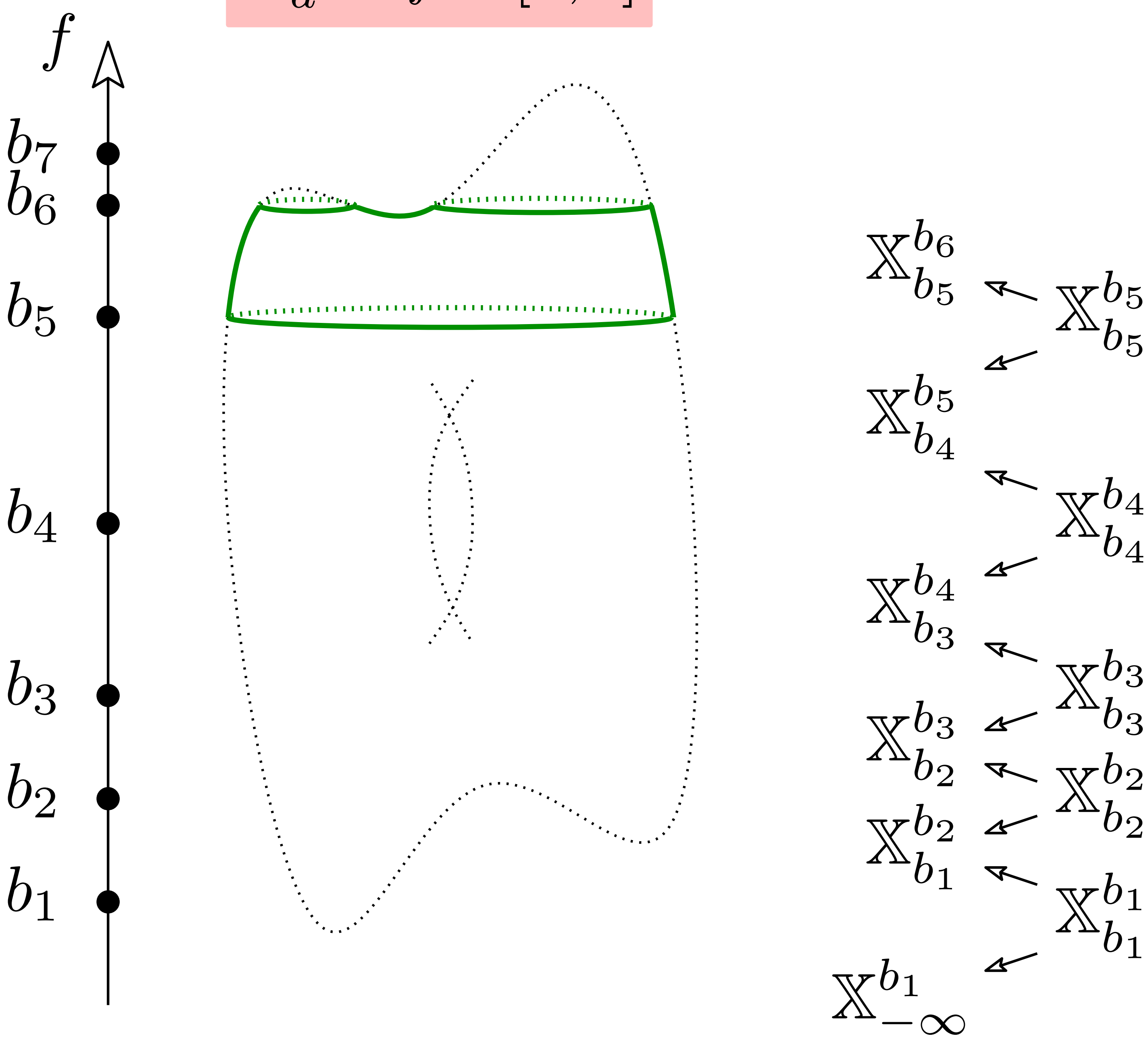
$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$



[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

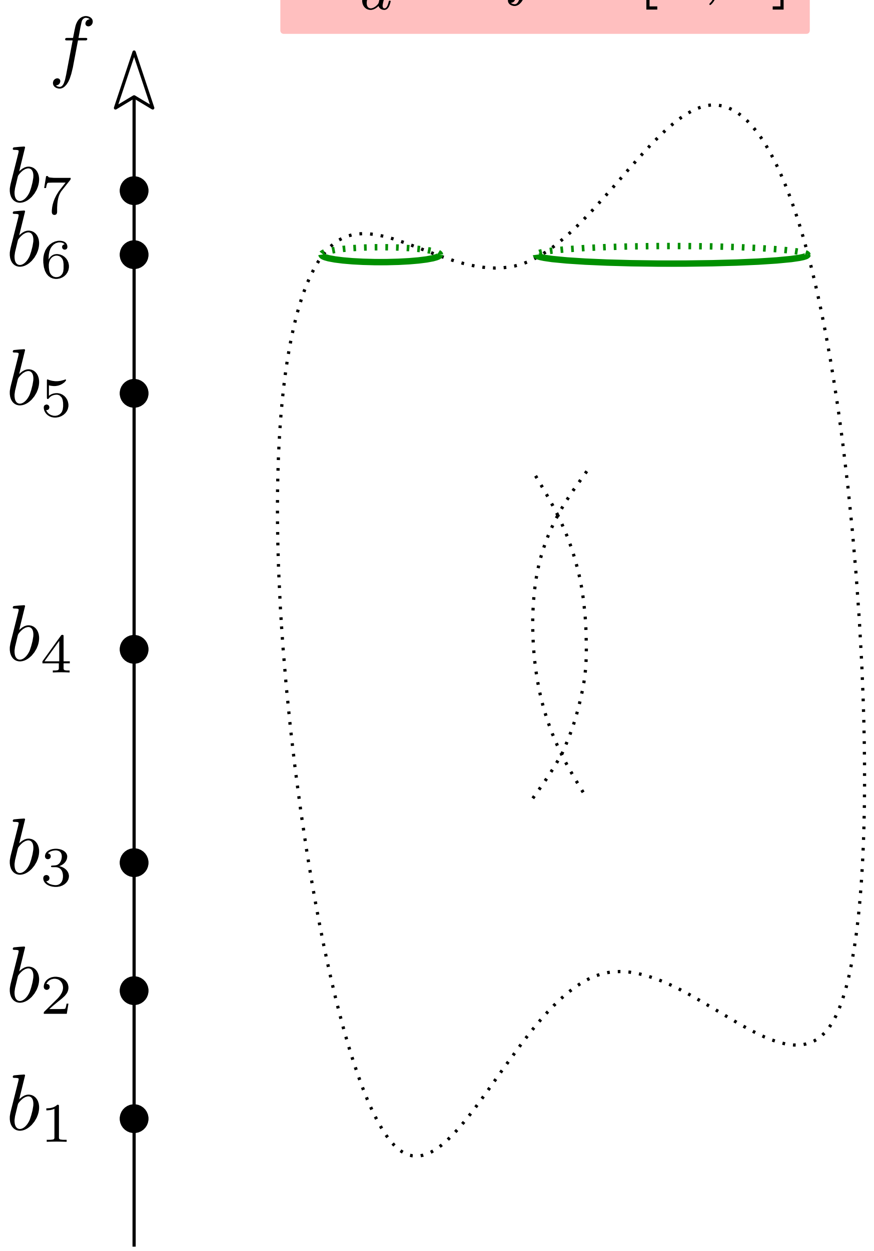
$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$



[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

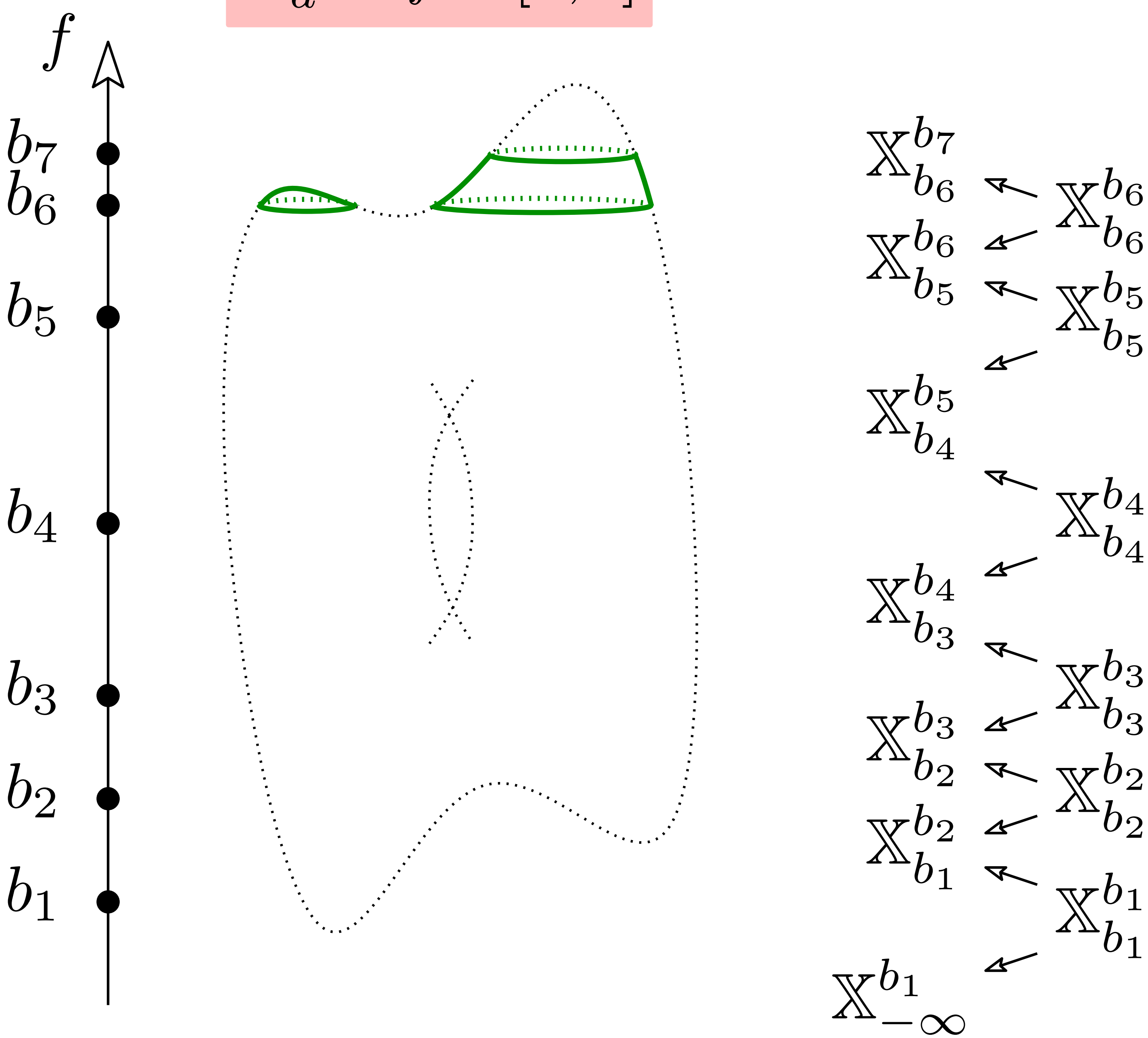


$$\mathbb{X}_{-\infty}^{b_1} \leftarrow \mathbb{X}_{b_1}^{b_1} \rightarrow \mathbb{X}_{b_1}^{b_2} \leftarrow \mathbb{X}_{b_2}^{b_2} \rightarrow \mathbb{X}_{b_2}^{b_3} \leftarrow \mathbb{X}_{b_3}^{b_3} \rightarrow \mathbb{X}_{b_3}^{b_4} \leftarrow \mathbb{X}_{b_4}^{b_4} \rightarrow \mathbb{X}_{b_4}^{b_5} \leftarrow \mathbb{X}_{b_5}^{b_5} \rightarrow \mathbb{X}_{b_5}^{b_6} \leftarrow \mathbb{X}_{b_6}^{b_6}$$

[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

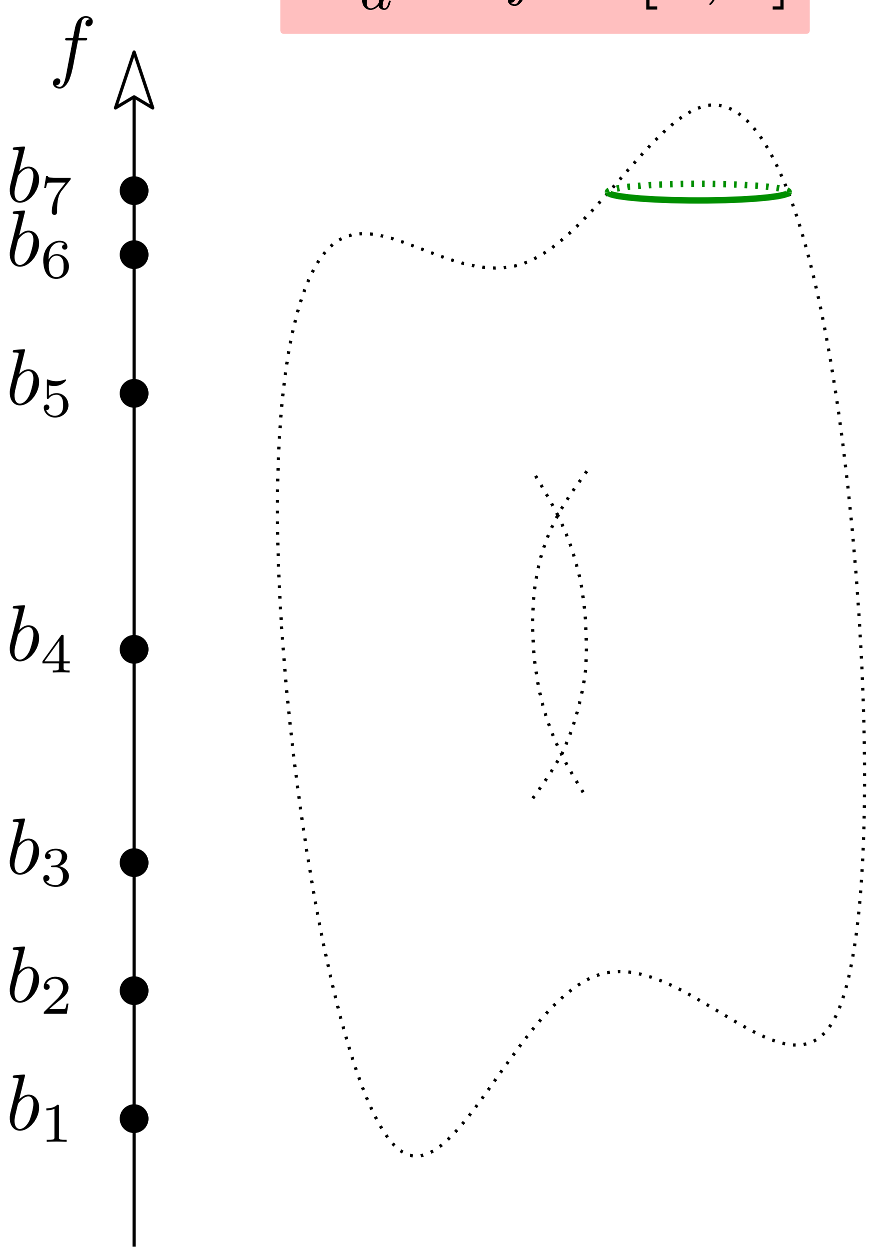
$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

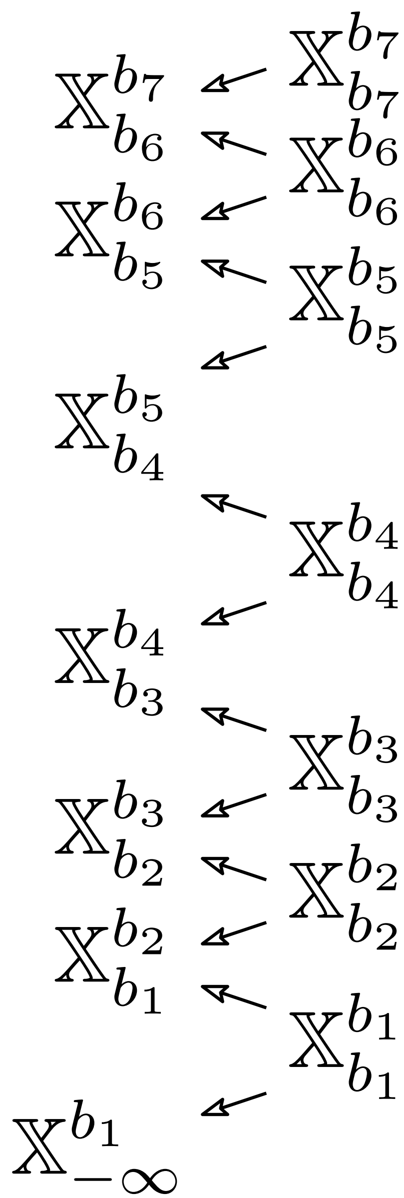


[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

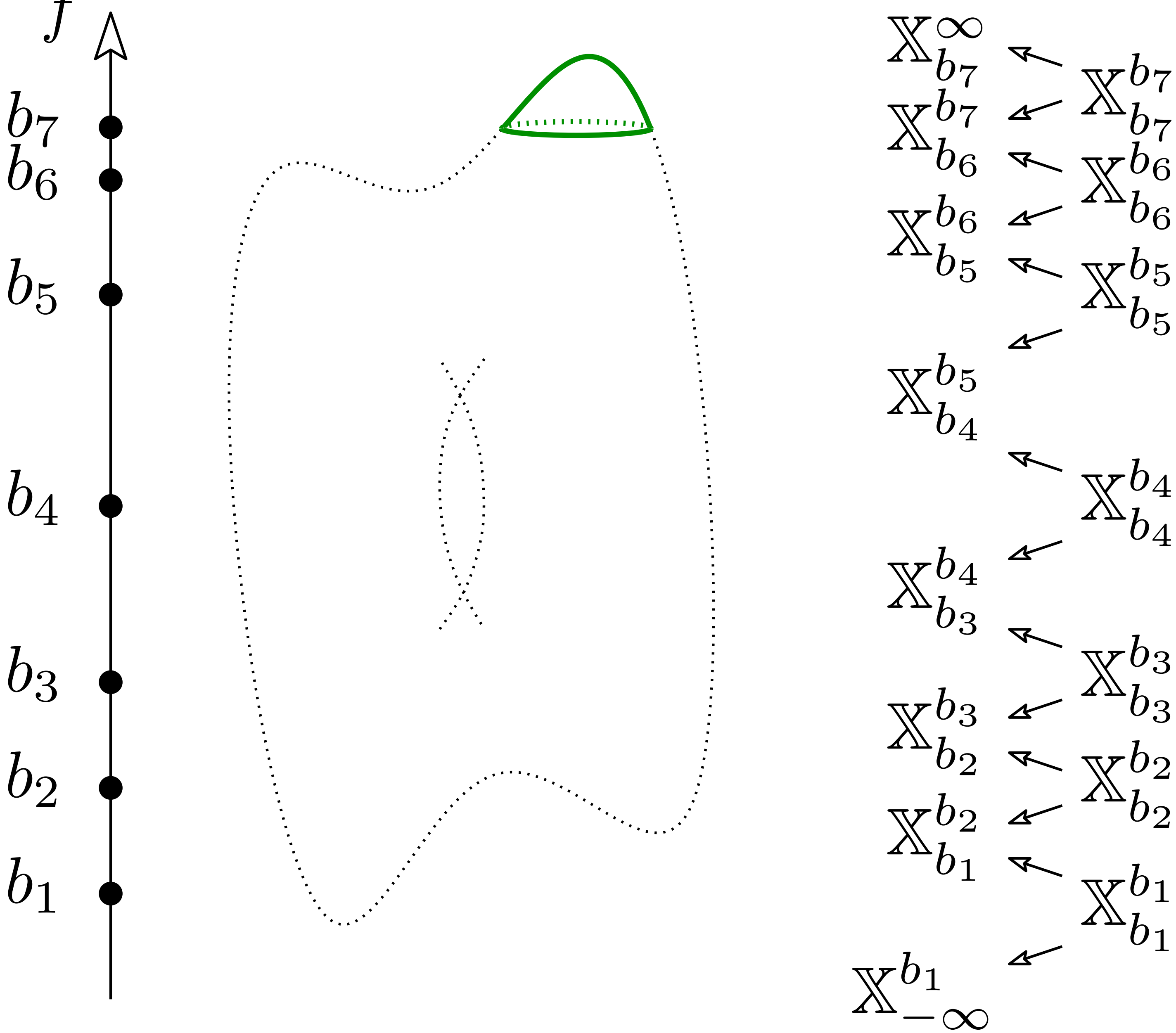




[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$

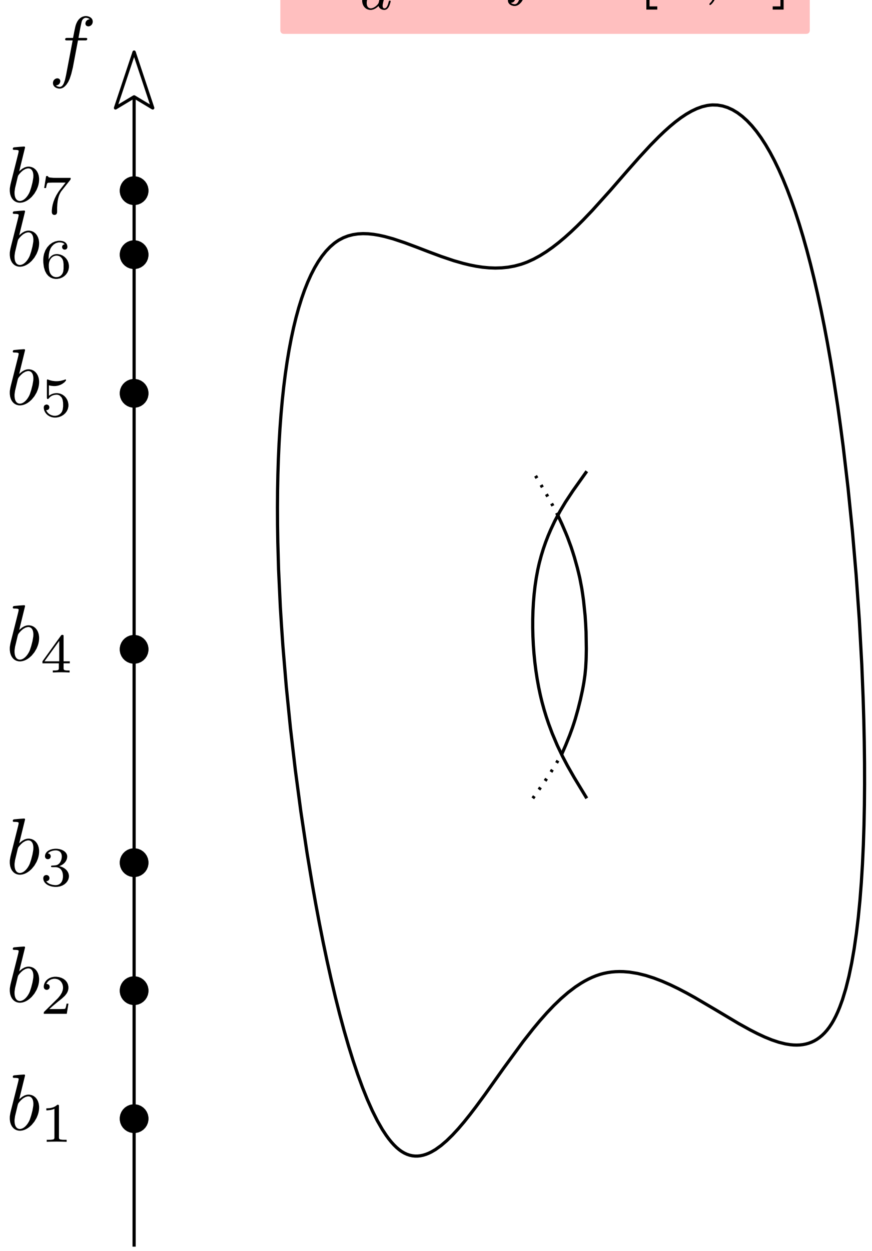


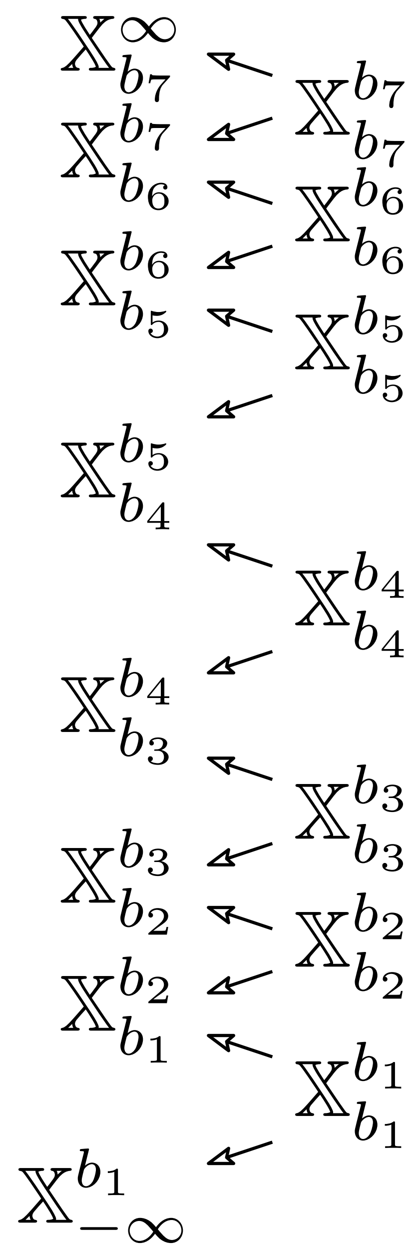
[Carlsson, de Silva, Morozov '09]

Levelset Zigzag

$f : \mathbb{X} \to \mathbb{R}$

$\mathbb{X}_a^b = f^{-1}[a, b]$





[Carlsson, de Silva, Morozov '09]

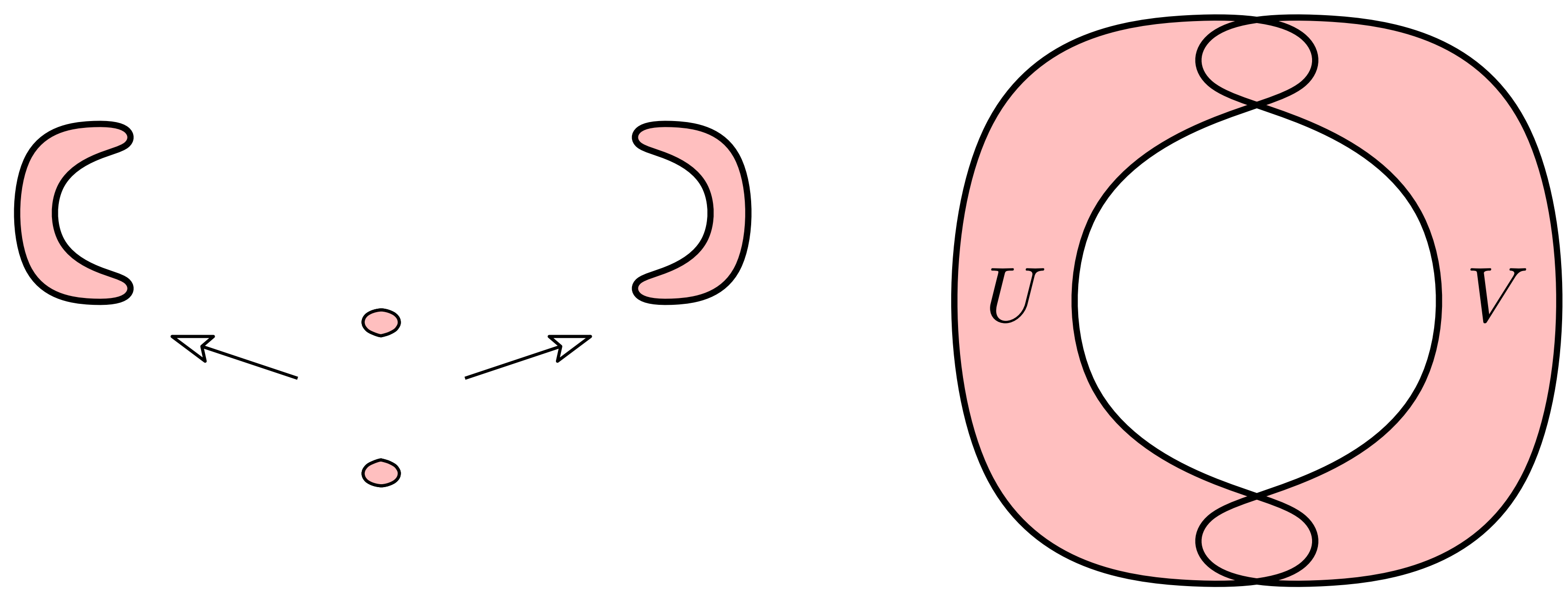
Mayer-Vietoris Diamond Principle

$$\begin{array}{ccccccccccccc}
\mathbb{X}_1 & \longrightarrow & \cdots & \longleftarrow & \mathbb{X}_{k-2} & \longrightarrow & \mathbb{U} & & & & \mathbb{V} & \longleftarrow & \mathbb{X}_{k+2} & \longrightarrow & \cdots & \longleftarrow & \mathbb{X}_n \\
 & & & & & & & \nwarrow & & \nearrow & & & & & & & \\
 & & & & & & & & \mathbb{U} \cap \mathbb{V} & & & & & & & &
\end{array}$$

Mayer-Vietoris Diamond Principle

$$\mathbb{X}_1 \longrightarrow \cdots \longleftarrow \mathbb{X}_{k-2} \longrightarrow \mathbb{U} \qquad\qquad \mathbb{V} \longleftarrow \mathbb{X}_{k+2} \longrightarrow \cdots \longleftarrow \mathbb{X}_n$$

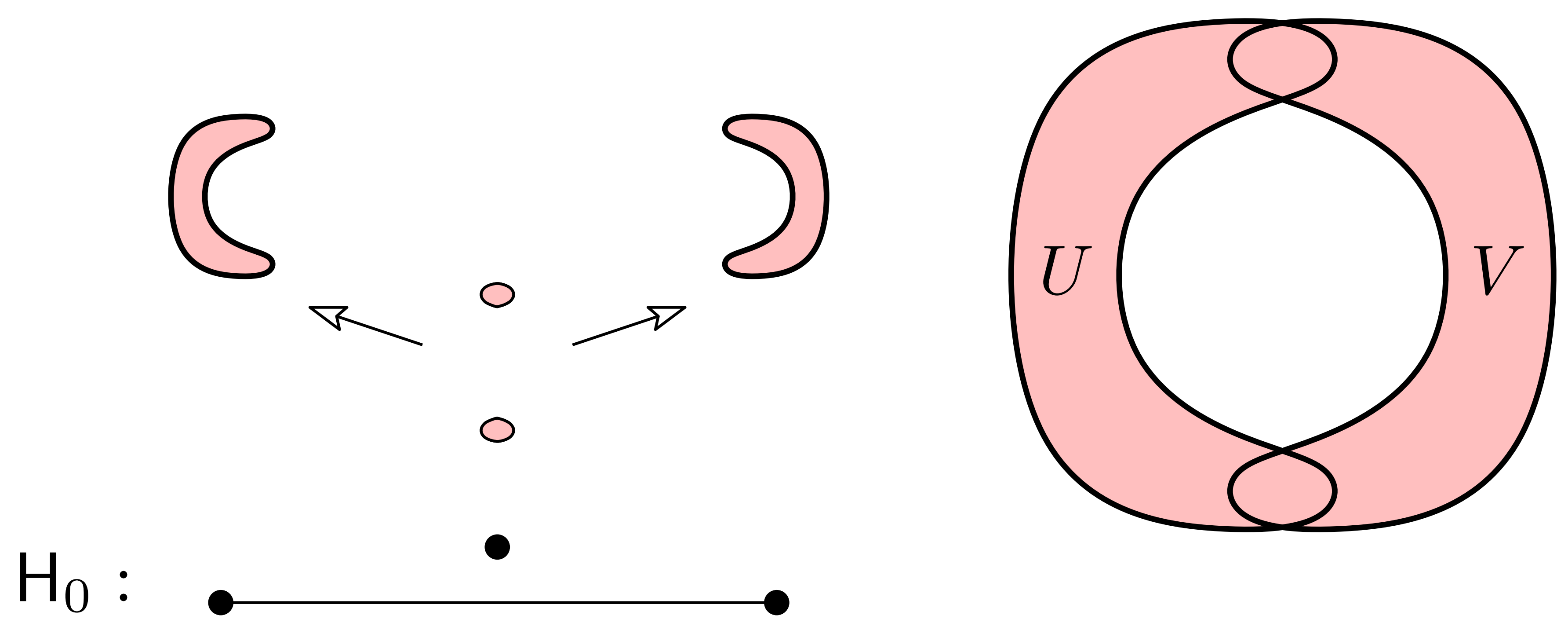
$$\mathbb{U} \longleftarrow \mathbb{U} \cap \mathbb{V} \longrightarrow \mathbb{V}$$



Mayer-Vietoris Diamond Principle

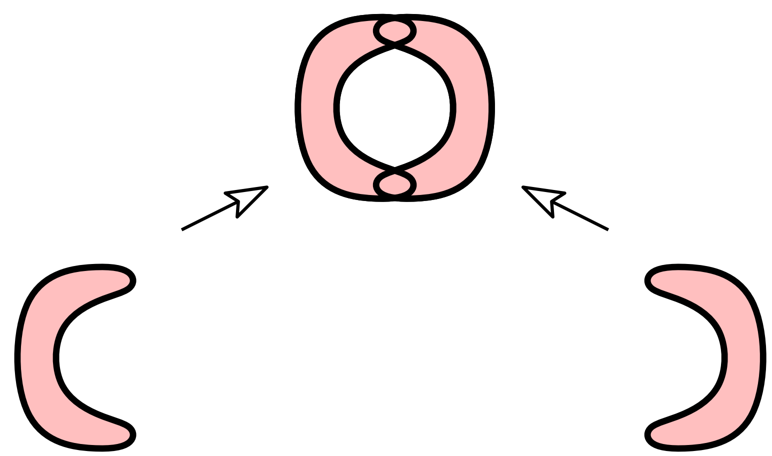
$$\mathbb{X}_1 \longrightarrow \cdots \longleftarrow \mathbb{X}_{k-2} \longrightarrow \mathbb{U} \qquad \mathbb{V} \longleftarrow \mathbb{X}_{k+2} \longrightarrow \cdots \longleftarrow \mathbb{X}_n$$

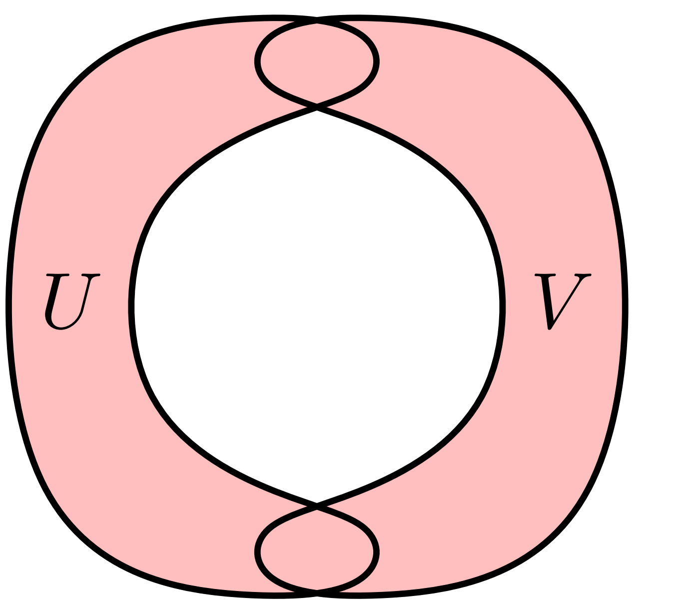
$$\mathbb{U} \nwarrow \mathbb{U} \cap \mathbb{V} \nearrow \mathbb{V}$$



Mayer-Vietoris Diamond Principle

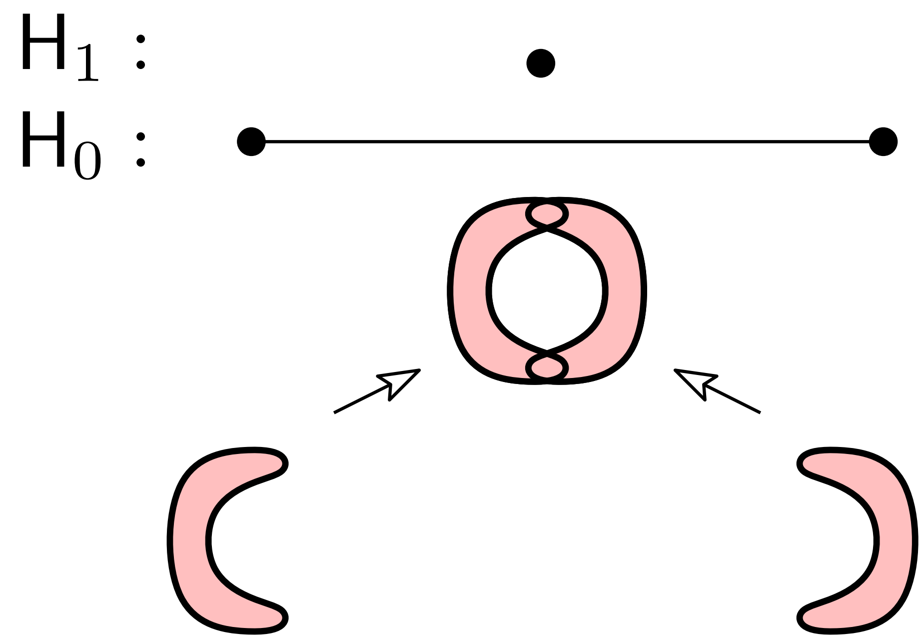
$$
\begin{array}{ccccccccccc}
 & & & & & \mathbb{U} \cup \mathbb{V} & & & & & \\
 & & & & \nearrow & & \nwarrow & & & & \\
\mathbb{X}_1 & \longrightarrow & \cdots & \longleftarrow & \mathbb{X}_{k-2} \longrightarrow \mathbb{U} & & \mathbb{V} \longleftarrow \mathbb{X}_{k+2} & \longrightarrow & \cdots & \longleftarrow & \mathbb{X}_n
\end{array}
$$

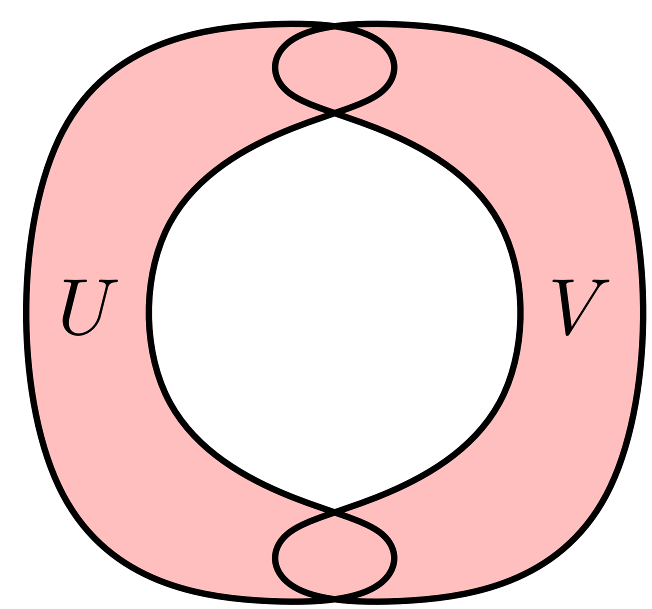




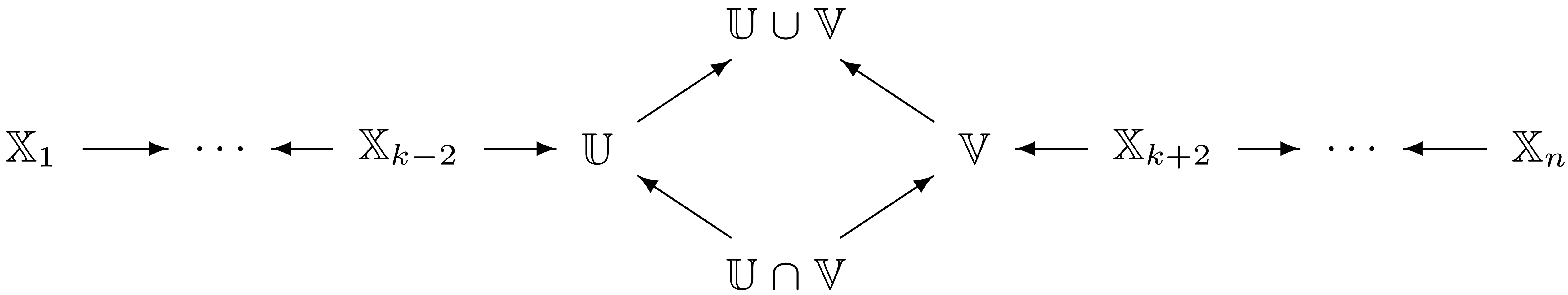
Mayer-Vietoris Diamond Principle

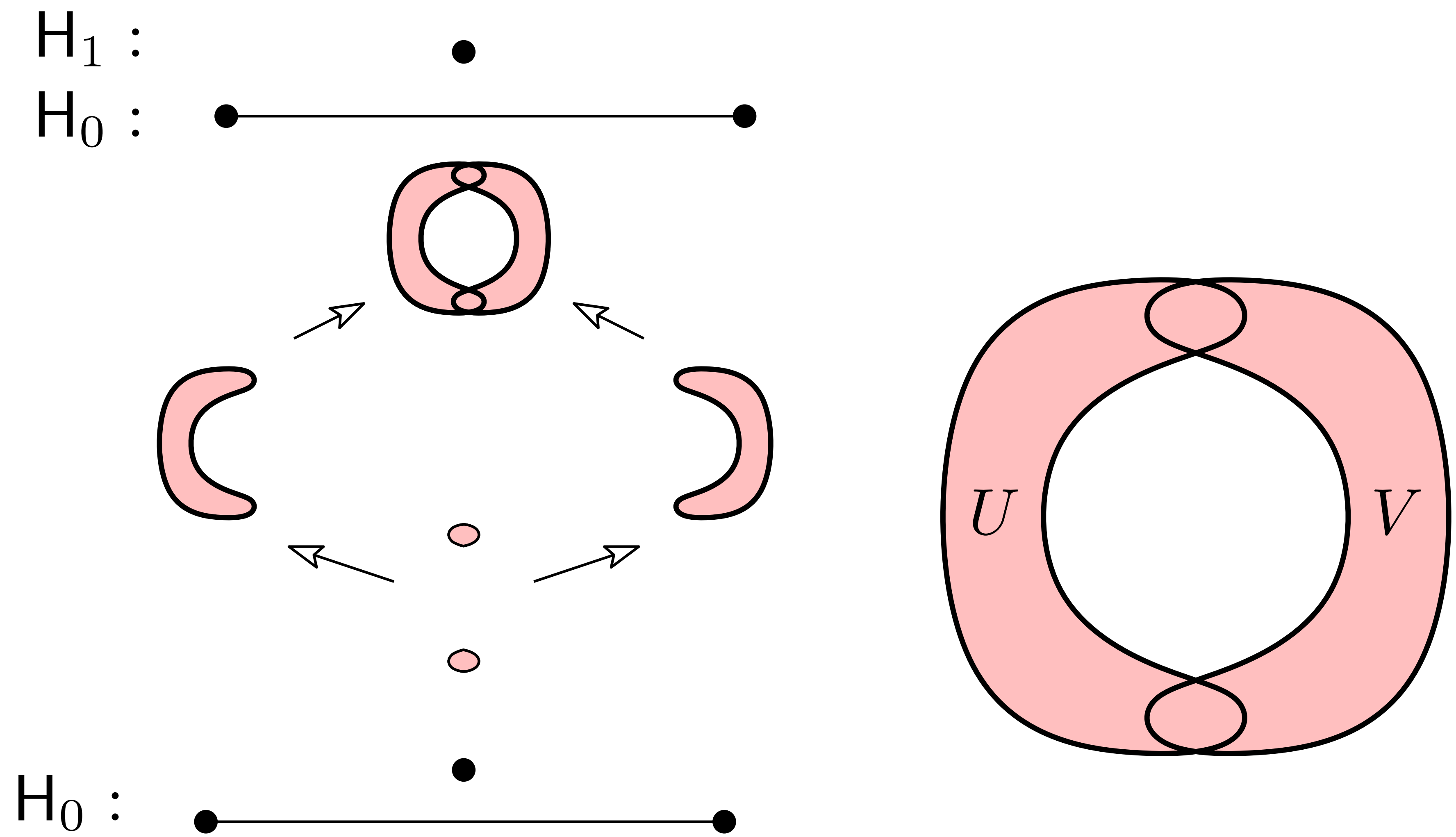
$$\mathbb{X}_1 \longrightarrow \cdots \longleftarrow \mathbb{X}_{k-2} \longrightarrow \mathbb{U} \longrightarrow \mathbb{U} \cup \mathbb{V} \longleftarrow \mathbb{V} \longleftarrow \mathbb{X}_{k+2} \longrightarrow \cdots \longleftarrow \mathbb{X}_n$$



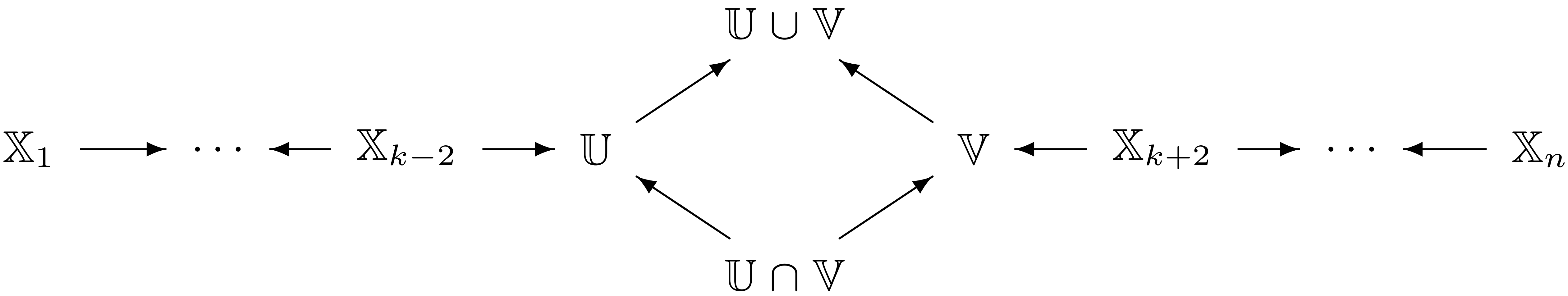


Mayer-Vietoris Diamond Principle





Mayer-Vietoris Diamond Principle

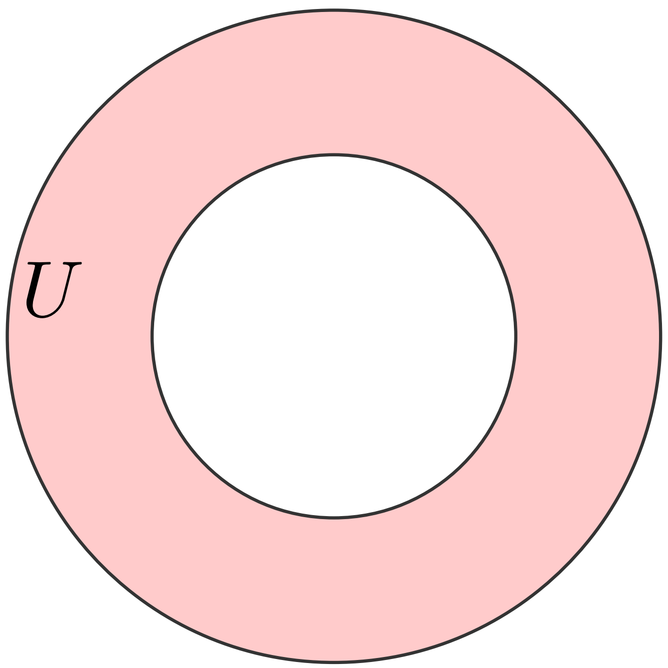


Theorem: There is a bijection between persistence intervals associated with two zigzag modules differing by a Mayer-Vietoris diamond.

[Carlsson, de Silva '10]

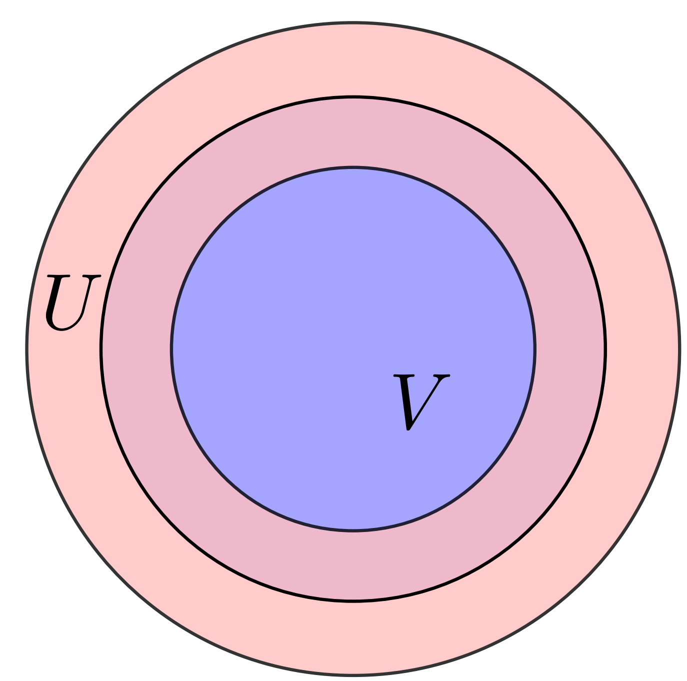
Mayer-Vietoris Diamond Principle

$$\begin{array}{ccccccccccccc}
\mathbb{X}_1 & \longrightarrow & \cdots & \longleftarrow & \mathbb{X}_{k-2} & \longrightarrow & \mathbb{U} & & \mathbb{V} & \longleftarrow & \mathbb{X}_{k+2} & \longrightarrow & \cdots & \longleftarrow & \mathbb{X}_n \\
 & & & & & & & \nwarrow \quad \nearrow & & & & & & & \\
 & & & & & & & \mathbb{U} \cap \mathbb{V} & & & & & & &
\end{array}$$



Mayer-Vietoris Diamond Principle

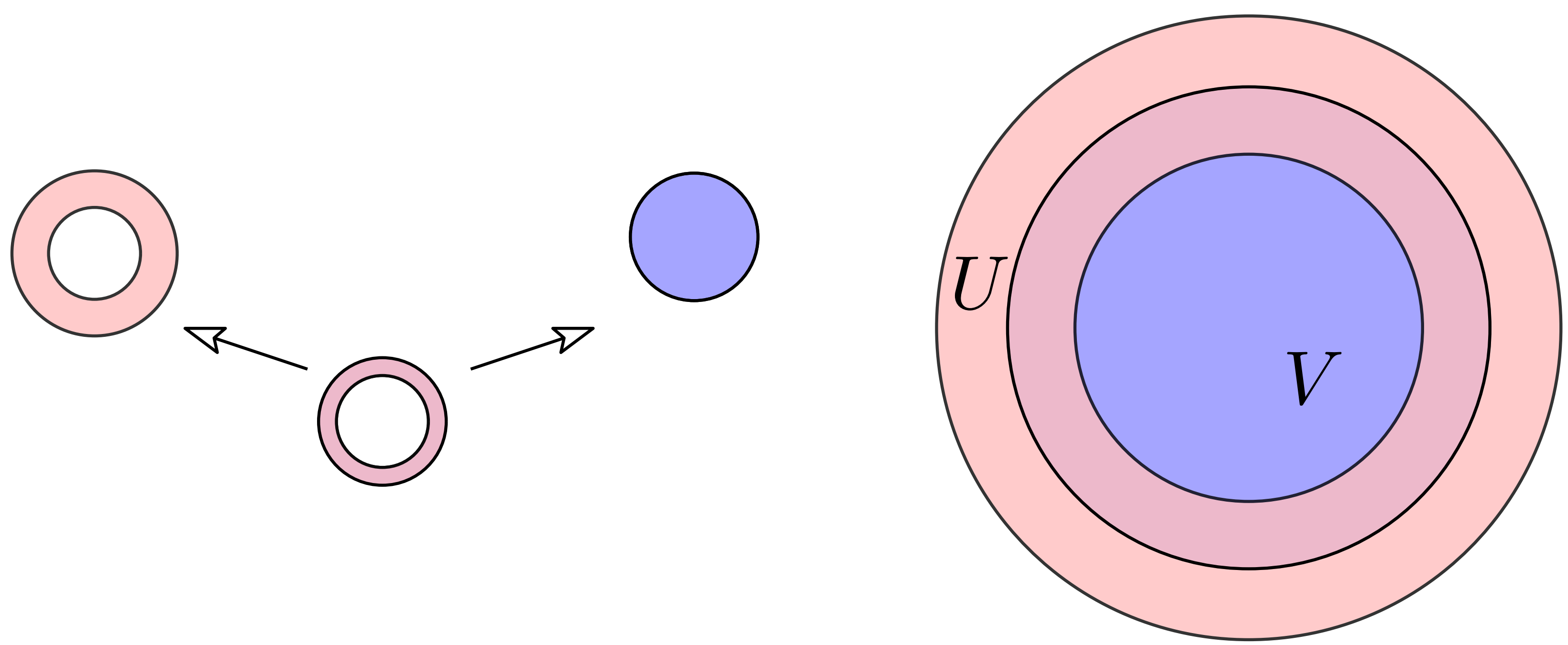
$$\mathbb{X}_1 \longrightarrow \cdots \longleftarrow \mathbb{X}_{k-2} \longrightarrow \mathbb{U} \longleftarrow \mathbb{U} \cap \mathbb{V} \longrightarrow \mathbb{V} \longleftarrow \mathbb{X}_{k+2} \longrightarrow \cdots \longleftarrow \mathbb{X}_n$$



Mayer-Vietoris Diamond Principle

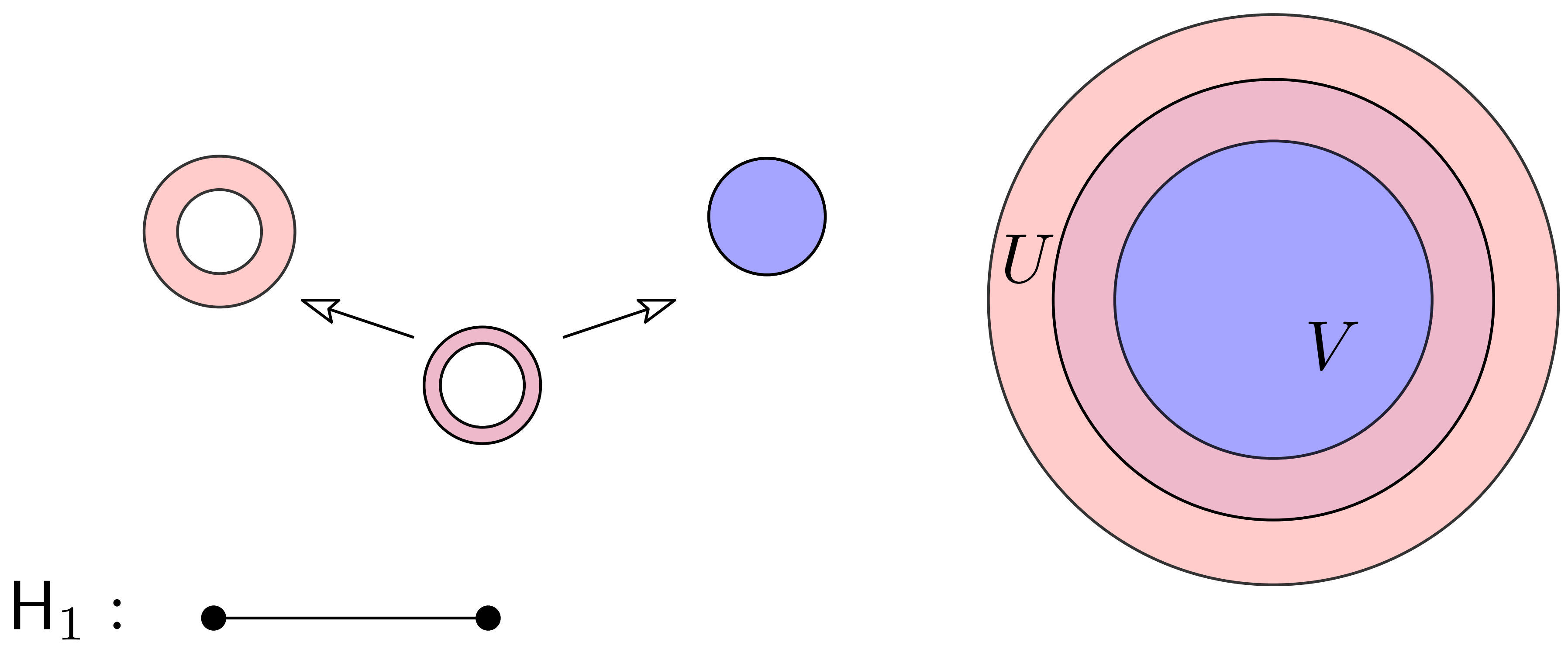
$$\mathbb{X}_1 \longrightarrow \cdots \longleftarrow \mathbb{X}_{k-2} \longrightarrow \mathbb{U} \qquad\qquad \mathbb{V} \longleftarrow \mathbb{X}_{k+2} \longrightarrow \cdots \longleftarrow \mathbb{X}_n$$

$$\mathbb{U} \longleftarrow \mathbb{U} \cap \mathbb{V} \longrightarrow \mathbb{V}$$



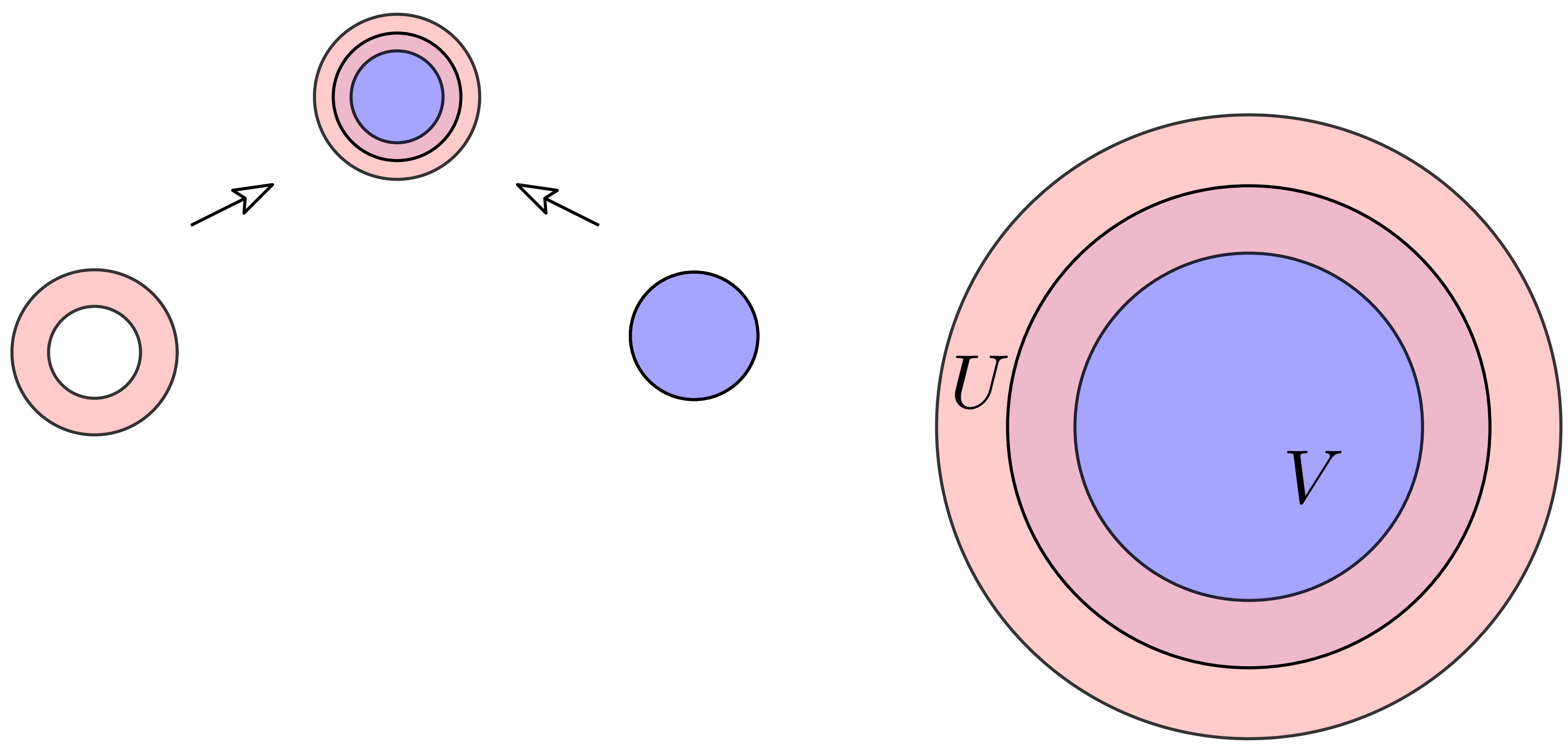
Mayer-Vietoris Diamond Principle

$$\mathbb{X}_1 \longrightarrow \cdots \longleftarrow \mathbb{X}_{k-2} \longrightarrow \mathbb{U} \longleftarrow \mathbb{U} \cap \mathbb{V} \longrightarrow \mathbb{V} \longleftarrow \mathbb{X}_{k+2} \longrightarrow \cdots \longleftarrow \mathbb{X}_n$$



Mayer-Vietoris Diamond Principle

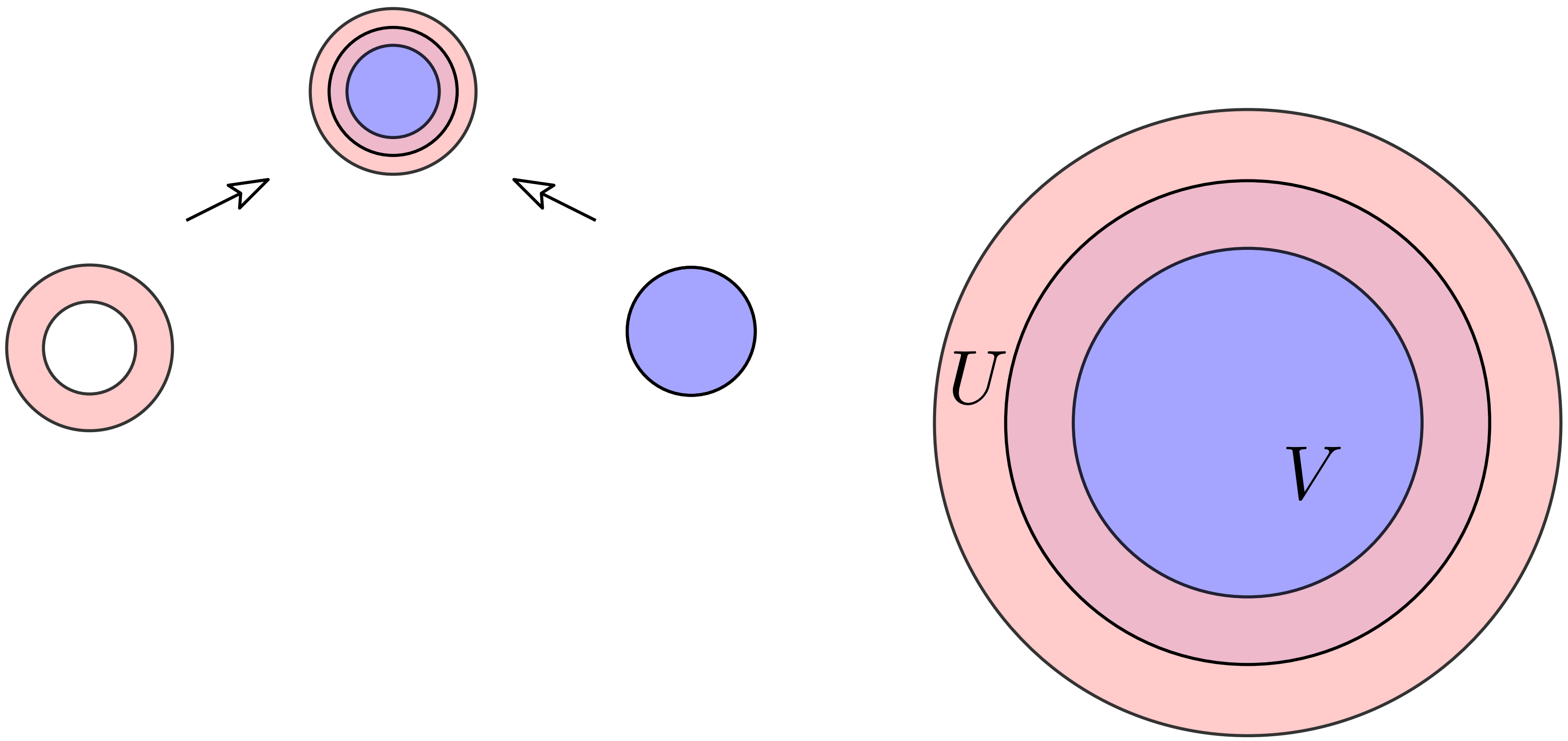
$$\mathbb{X}_1 \longrightarrow \cdots \longleftarrow \mathbb{X}_{k-2} \longrightarrow \mathbb{U} \longleftarrow \mathbb{U} \cap \mathbb{V} \longrightarrow \mathbb{V} \longleftarrow \mathbb{X}_{k+2} \longrightarrow \cdots \longleftarrow \mathbb{X}_n$$



Mayer-Vietoris Diamond Principle

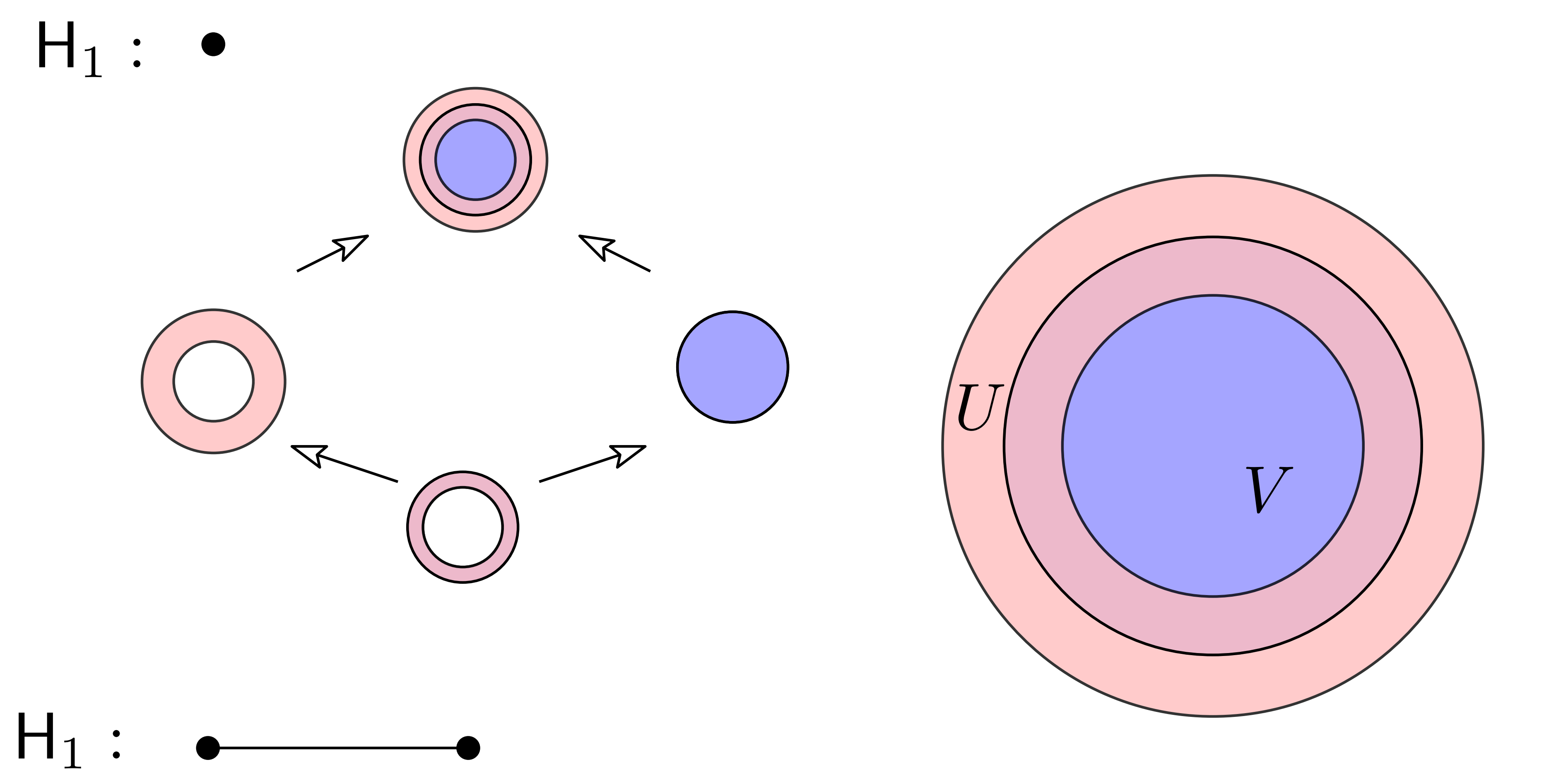
$$\mathbb{X}_1 \longrightarrow \cdots \longleftarrow \mathbb{X}_{k-2} \longrightarrow \mathbb{U} \longleftarrow \mathbb{U} \cap \mathbb{V} \longrightarrow \mathbb{V} \longleftarrow \mathbb{X}_{k+2} \longrightarrow \cdots \longleftarrow \mathbb{X}_n$$

H_1 : •

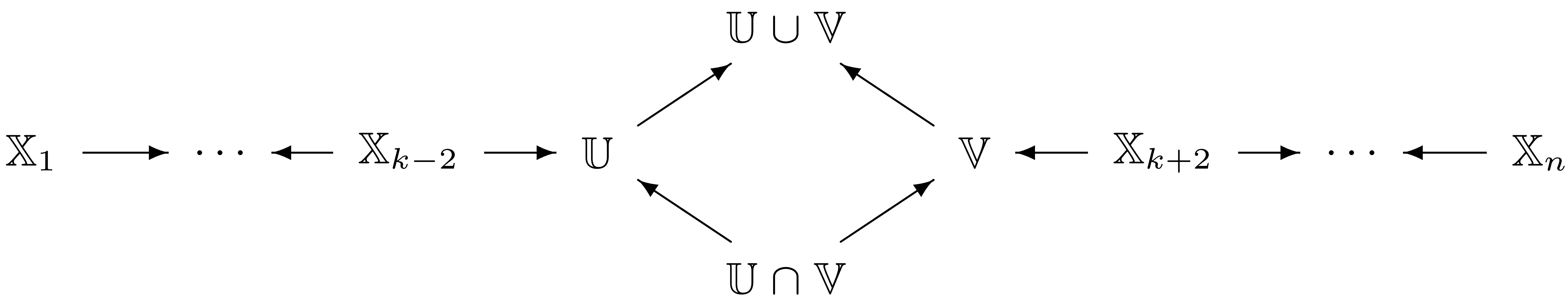


Mayer-Vietoris Diamond Principle

$$\begin{array}{ccccccccccccc} \mathbb{X}_1 & \longrightarrow & \cdots & \longleftarrow & \mathbb{X}_{k-2} & \longrightarrow & \mathbb{U} & & & & \mathbb{V} & \longleftarrow & \mathbb{X}_{k+2} & \longrightarrow & \cdots & \longleftarrow & \mathbb{X}_n \\ & & & & & & & \nwarrow & & \nearrow & & & & & & & \\ & & & & & & & & \mathbb{U} \cap \mathbb{V} & & & & & & & & \end{array}$$

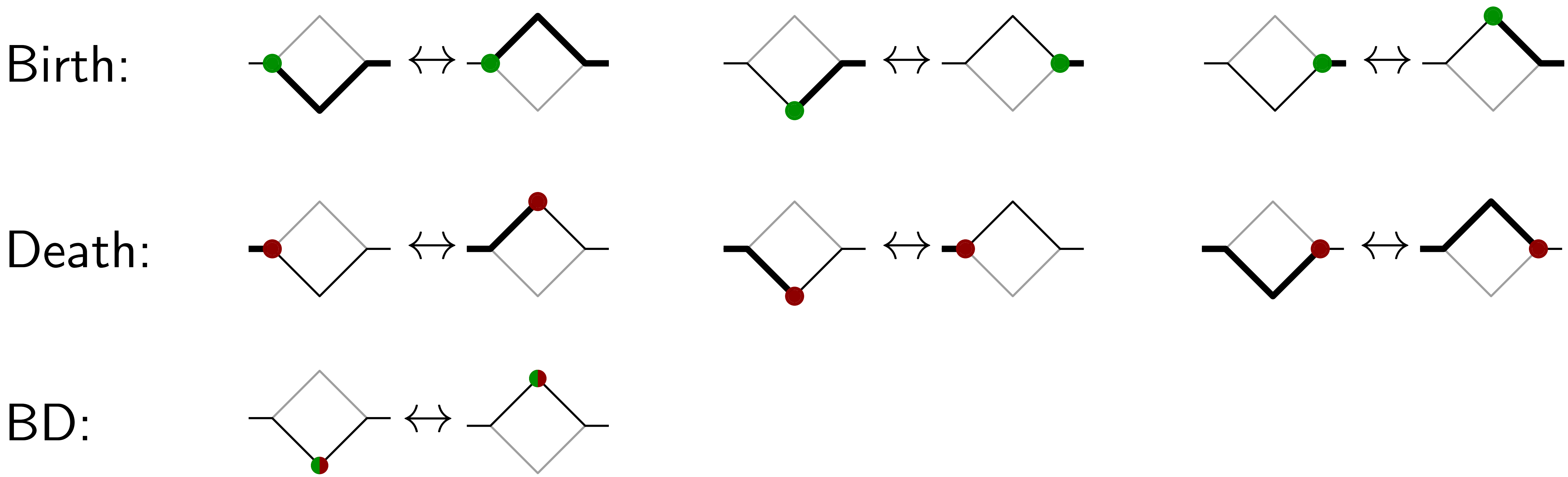


Mayer-Vietoris Diamond Principle

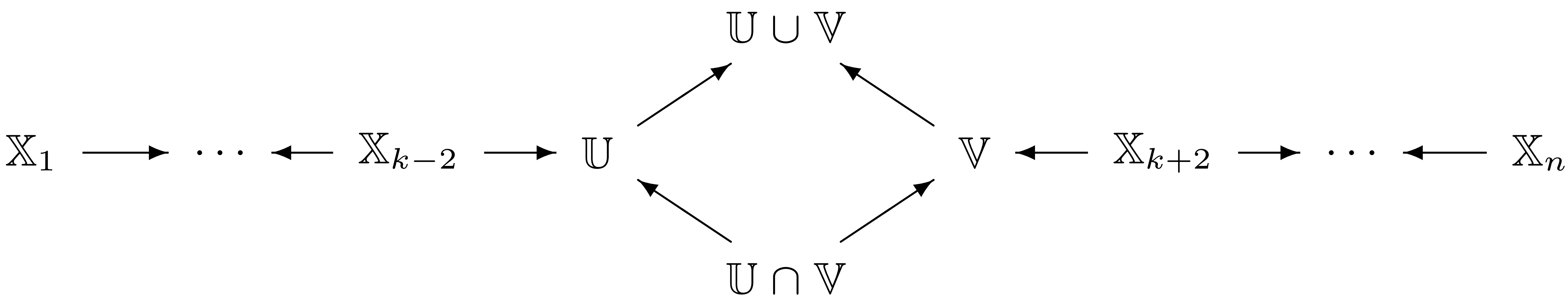


Theorem: There is a bijection between persistence intervals associated with two zigzag modules differing by a Mayer-Vietoris diamond.

[Carlsson, de Silva '10]



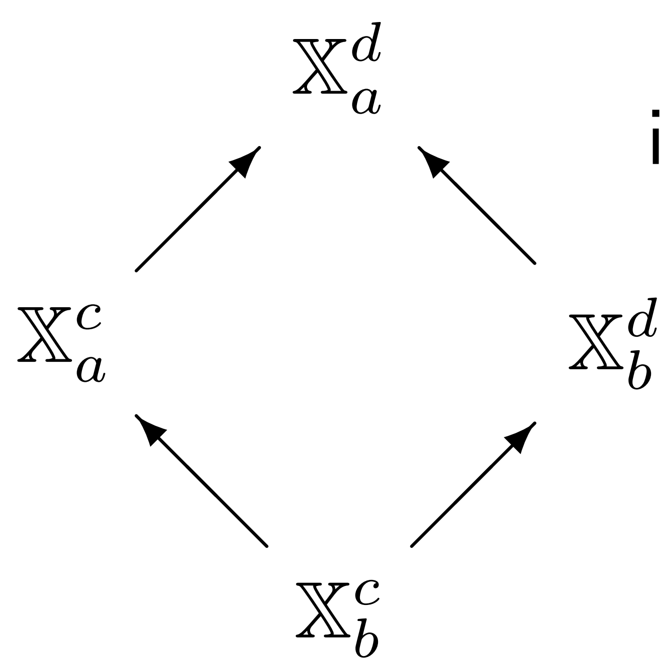
Mayer-Vietoris Diamond Principle



Theorem: There is a bijection between persistence intervals associated with two zigzag modules differing by a Mayer-Vietoris diamond.

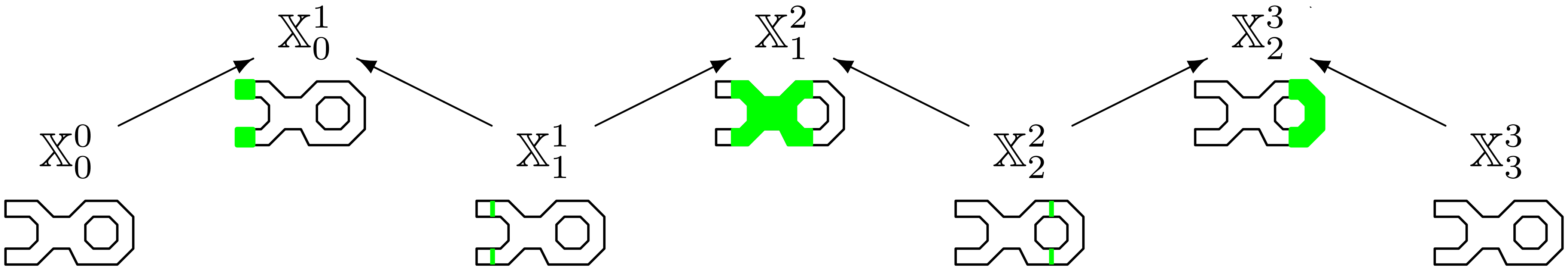
[Carlsson, de Silva '10]

$f : \mathbb{X} \to \mathbb{R}$
$\mathbb{X}_a^b = f^{-1}[a, b]$



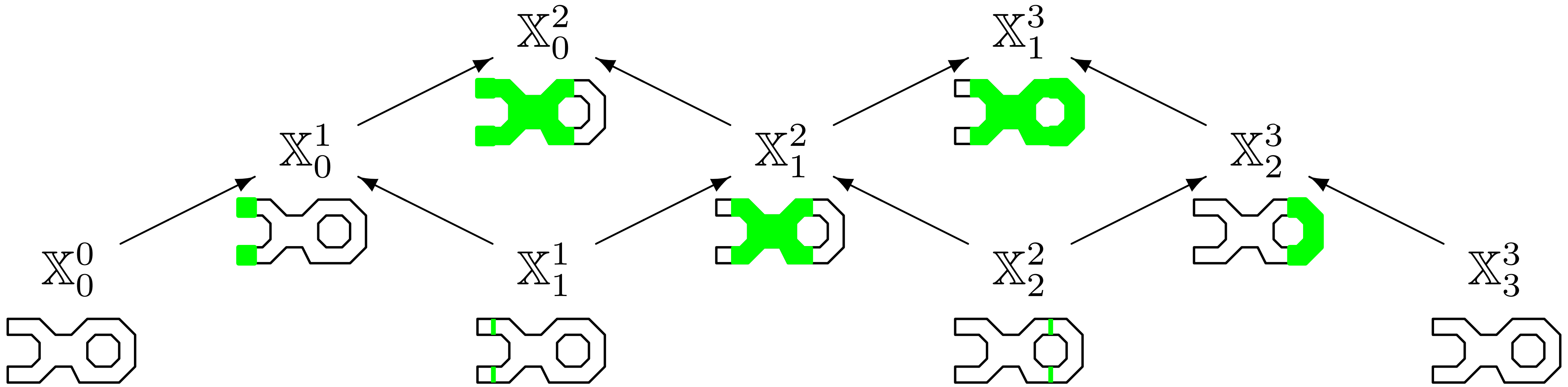
if $a < b < c < d$

Pyramid



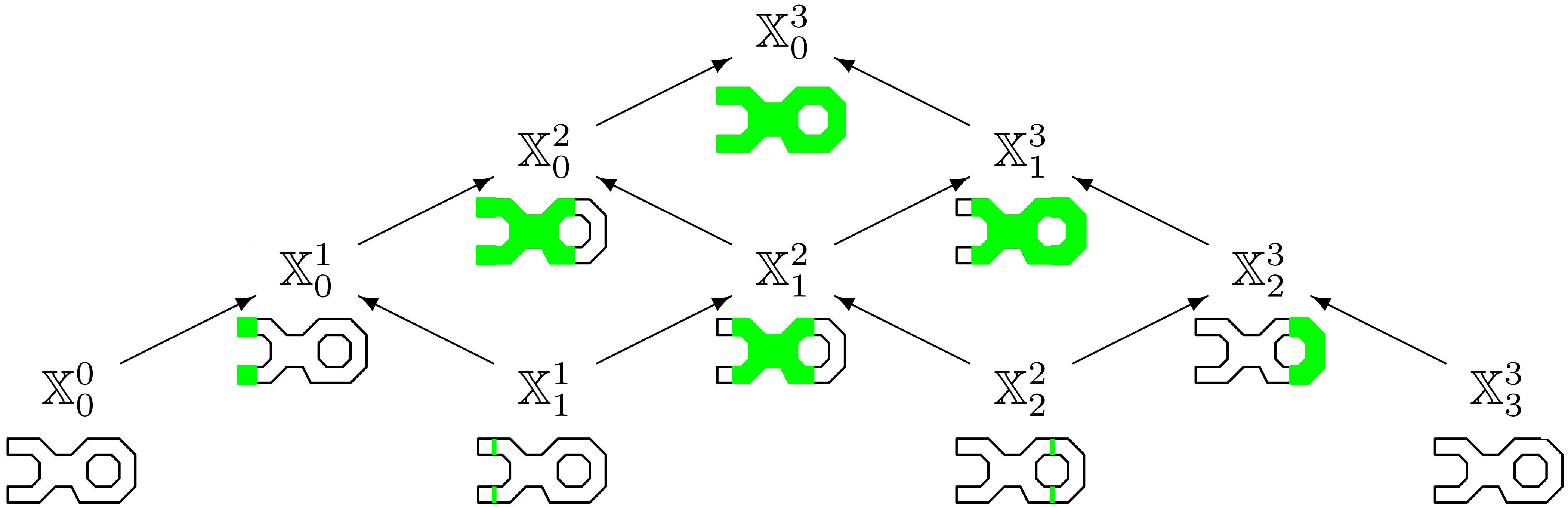
$\mathbb{X}_0^1$
$\mathbb{X}_1^2$
$\mathbb{X}_2^3$
$\mathbb{X}_0^0$
$\mathbb{X}_1^1$
$\mathbb{X}_2^2$
$\mathbb{X}_3^3$

Pyramid



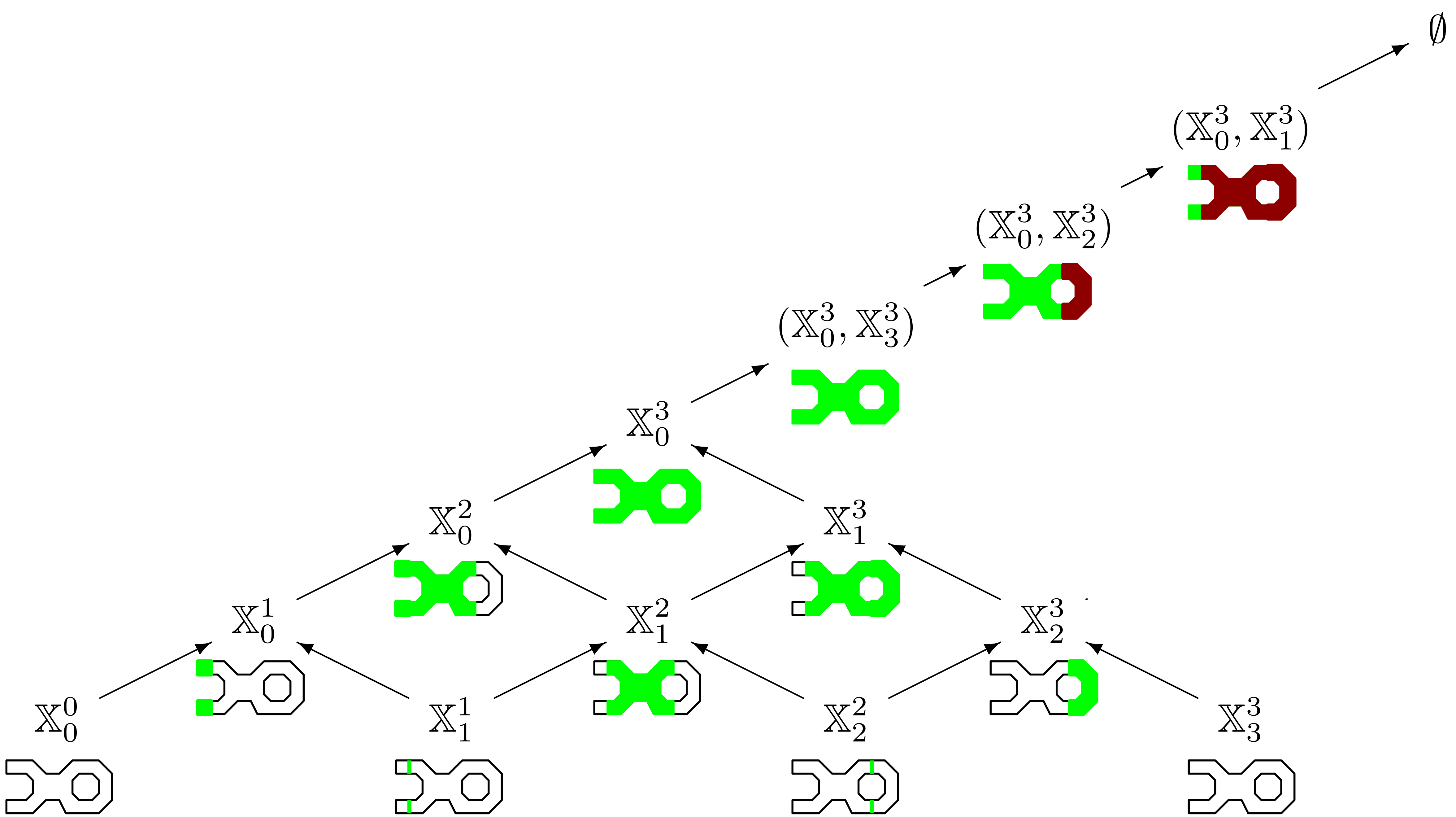
$\mathbb{X}_0^2$
$\mathbb{X}_1^3$
$\mathbb{X}_0^1$
$\mathbb{X}_1^2$
$\mathbb{X}_2^3$
$\mathbb{X}_0^0$
$\mathbb{X}_1^1$
$\mathbb{X}_2^2$
$\mathbb{X}_3^3$

Pyramid

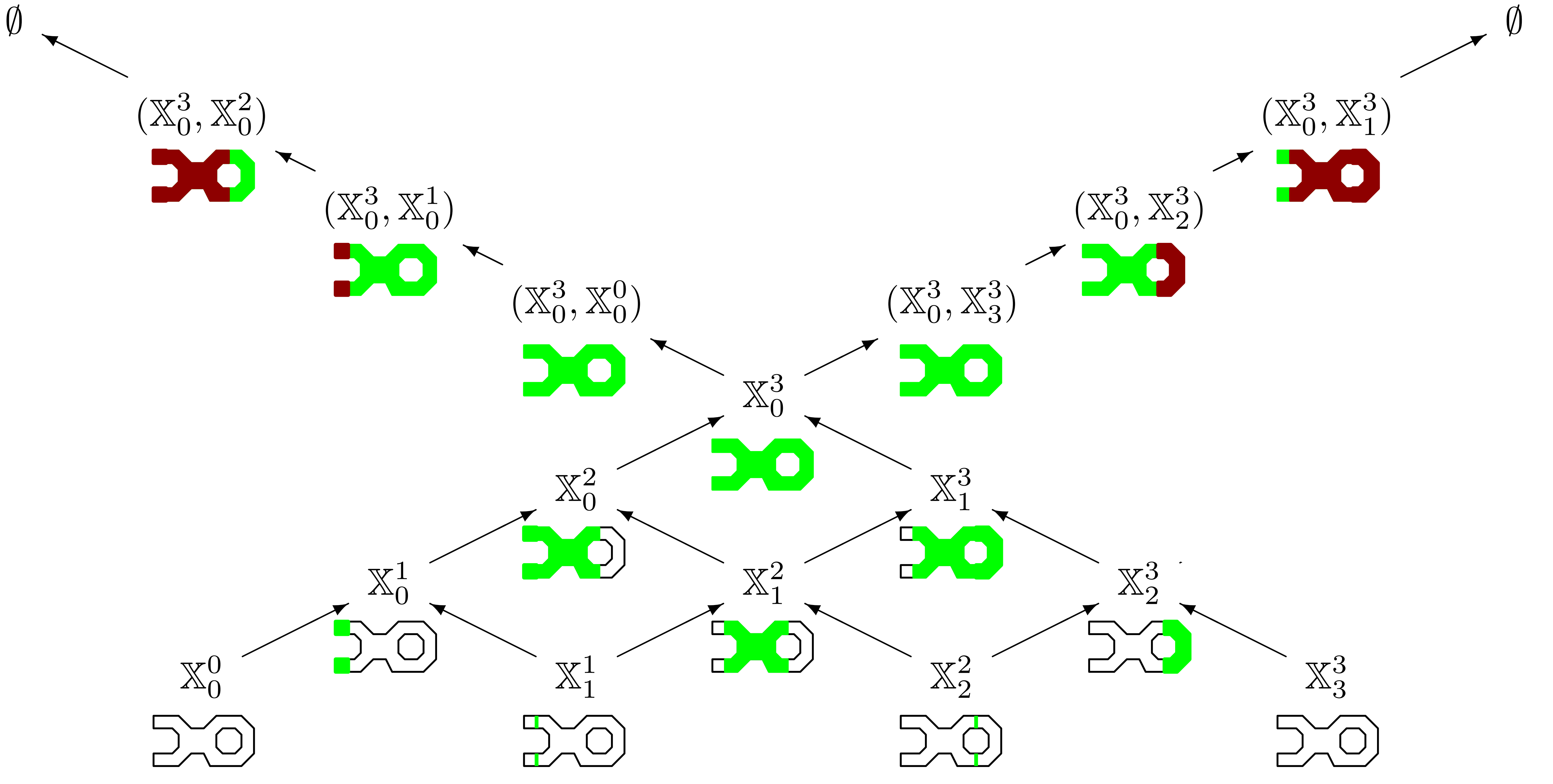


$\mathbb{X}_0^3$
$\mathbb{X}_0^2$
$\mathbb{X}_1^3$
$\mathbb{X}_0^1$
$\mathbb{X}_1^2$
$\mathbb{X}_2^3$
$\mathbb{X}_0^0$
$\mathbb{X}_1^1$
$\mathbb{X}_2^2$
$\mathbb{X}_3^3$

Pyramid

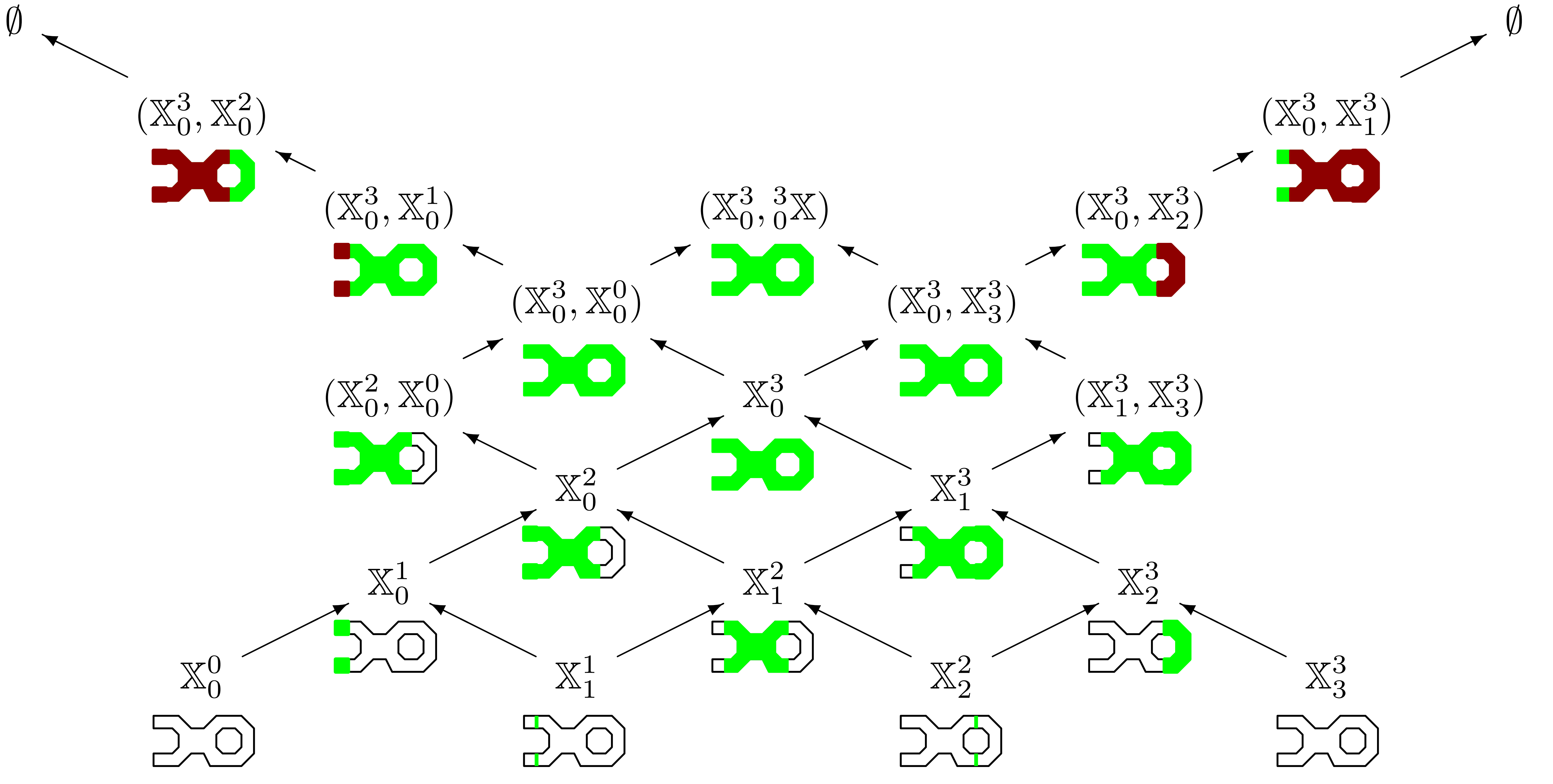

$\emptyset$
$(\mathbb{X}_0^3, \mathbb{X}_1^3)$
$(\mathbb{X}_0^3, \mathbb{X}_2^3)$
$(\mathbb{X}_0^3, \mathbb{X}_3^3)$
$\mathbb{X}_0^3$
$\mathbb{X}_0^2$
$\mathbb{X}_1^3$
$\mathbb{X}_0^1$
$\mathbb{X}_1^2$
$\mathbb{X}_2^3$
$\mathbb{X}_0^0$
$\mathbb{X}_1^1$
$\mathbb{X}_2^2$
$\mathbb{X}_3^3$

Pyramid



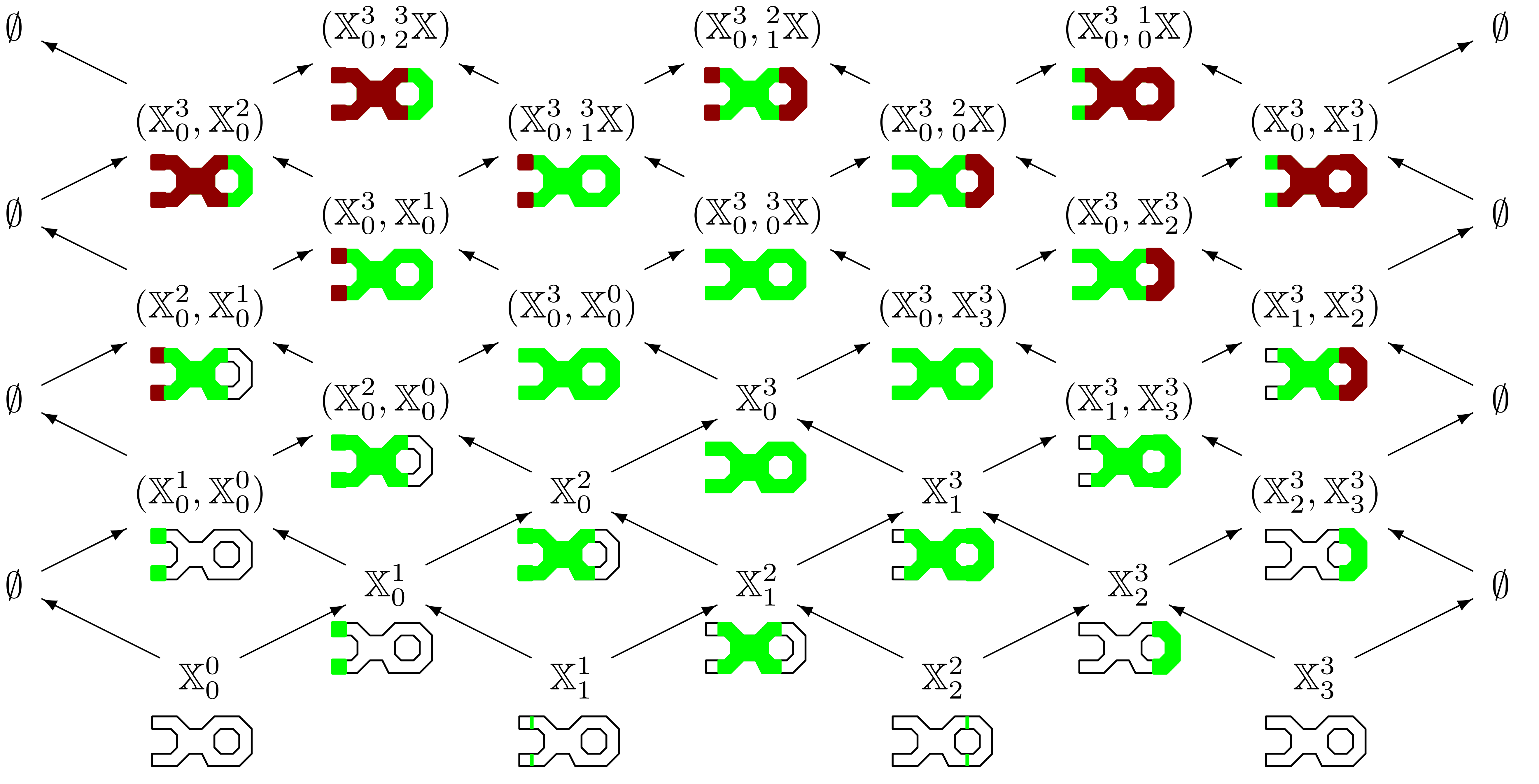
$\emptyset$
$\emptyset$
$(\mathbb{X}_0^3, \mathbb{X}_0^2)$
$(\mathbb{X}_0^3, \mathbb{X}_1^3)$
$(\mathbb{X}_0^3, \mathbb{X}_0^1)$
$(\mathbb{X}_0^3, \mathbb{X}_2^3)$
$(\mathbb{X}_0^3, \mathbb{X}_0^0)$
$(\mathbb{X}_0^3, \mathbb{X}_3^3)$
$\mathbb{X}_0^3$
$\mathbb{X}_0^2$
$\mathbb{X}_1^3$
$\mathbb{X}_0^1$
$\mathbb{X}_1^2$
$\mathbb{X}_2^3$
$\mathbb{X}_0^0$
$\mathbb{X}_1^1$
$\mathbb{X}_2^2$
$\mathbb{X}_3^3$

Pyramid



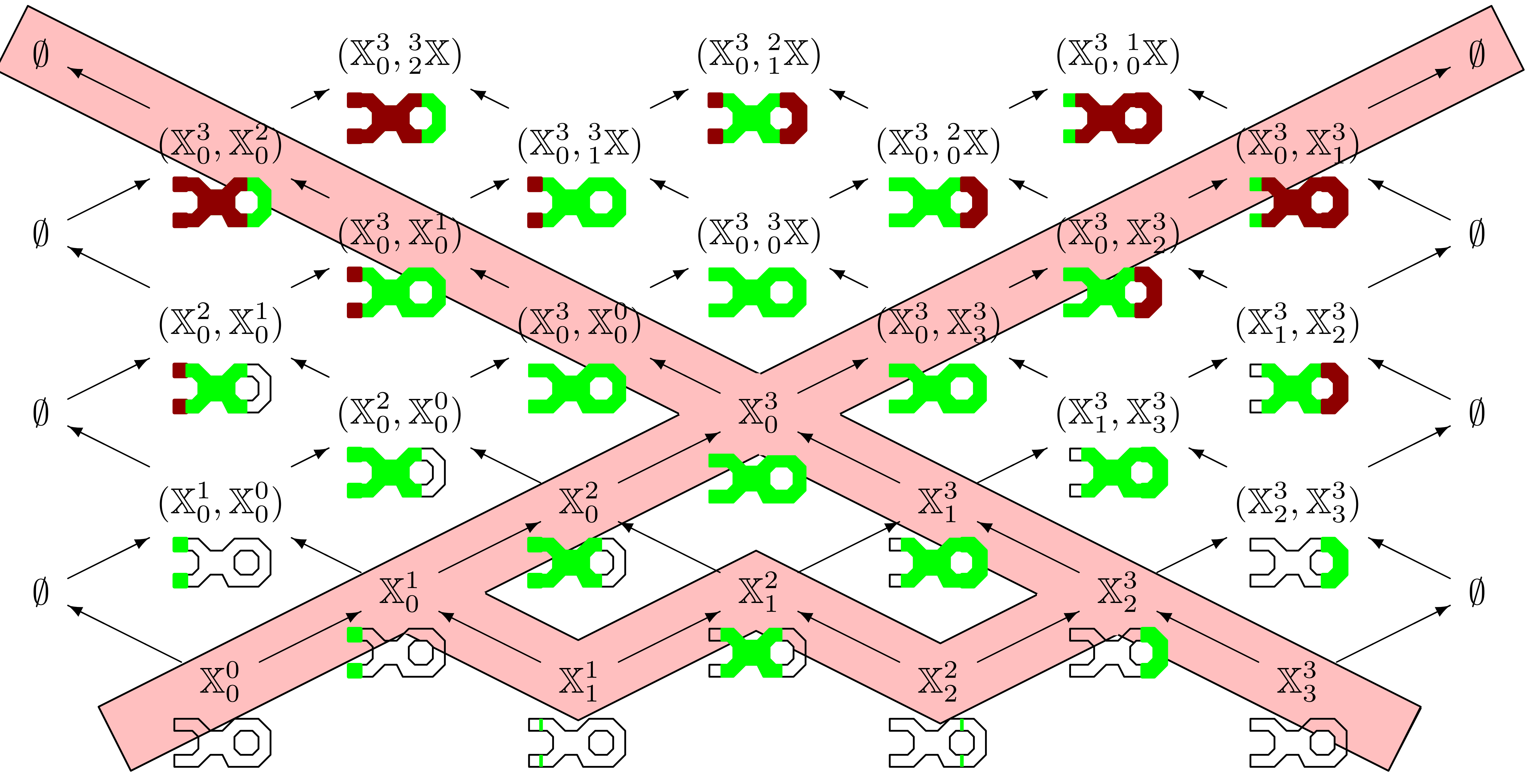
$\emptyset$
$\emptyset$
$(\mathbb{X}_0^3, \mathbb{X}_0^2)$
$(\mathbb{X}_0^3, \mathbb{X}_1^3)$
$(\mathbb{X}_0^3, \mathbb{X}_0^1)$
$(\mathbb{X}_0^3, {}_0^3\mathbb{X})$
$(\mathbb{X}_0^3, \mathbb{X}_2^3)$
$(\mathbb{X}_0^3, \mathbb{X}_0^0)$
$(\mathbb{X}_0^3, \mathbb{X}_3^3)$
$(\mathbb{X}_0^2, \mathbb{X}_0^0)$
$\mathbb{X}_0^3$
$(\mathbb{X}_1^3, \mathbb{X}_3^3)$
$\mathbb{X}_0^2$
$\mathbb{X}_1^3$
$\mathbb{X}_0^1$
$\mathbb{X}_1^2$
$\mathbb{X}_2^3$
$\mathbb{X}_0^0$
$\mathbb{X}_1^1$
$\mathbb{X}_2^2$
$\mathbb{X}_3^3$

Pyramid



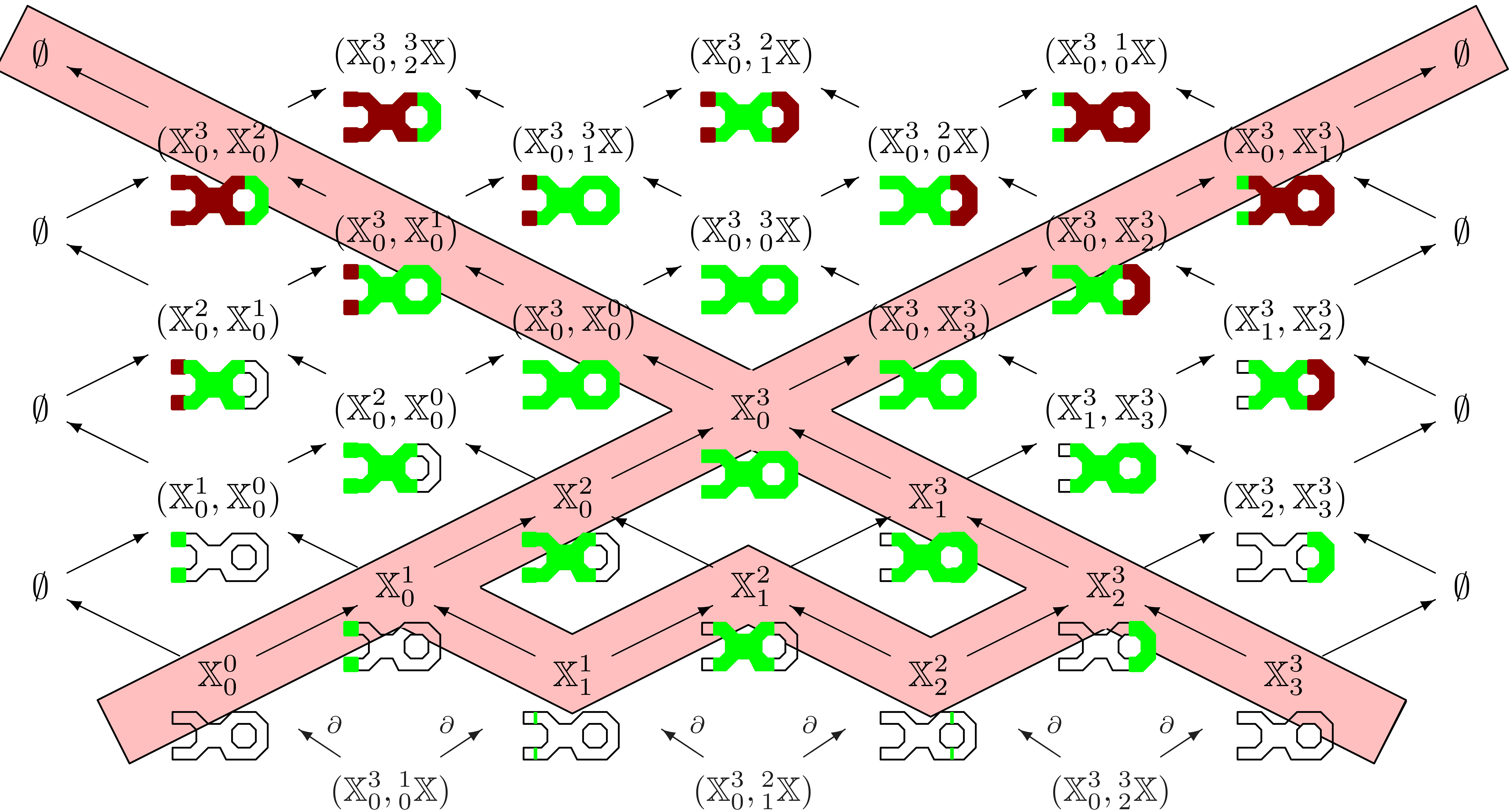
$${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Pyramid



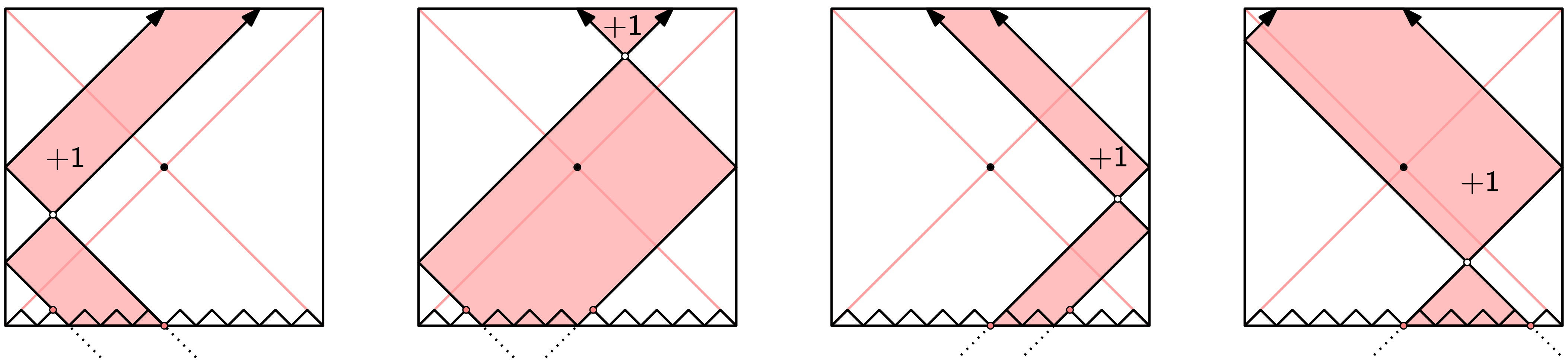
$${}^j_i\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

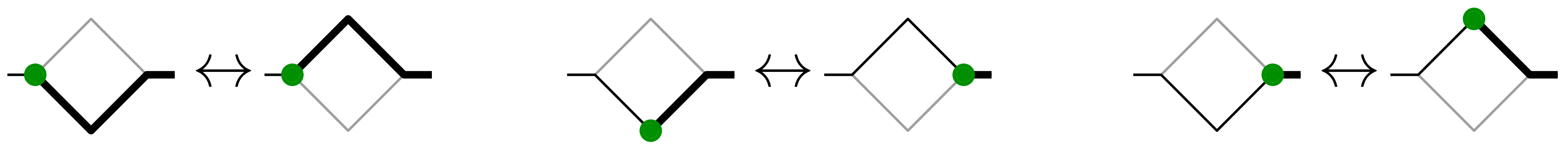
Pyramid

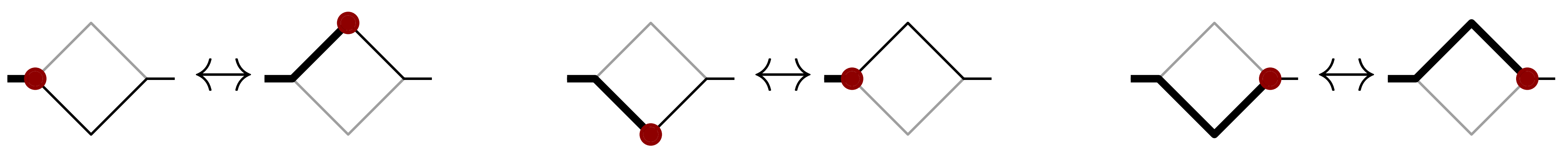


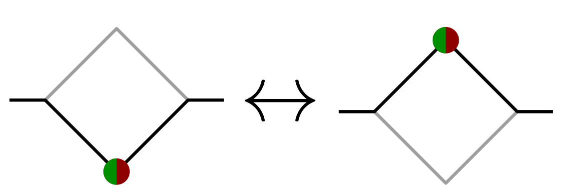
$${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Pyramid Transformation

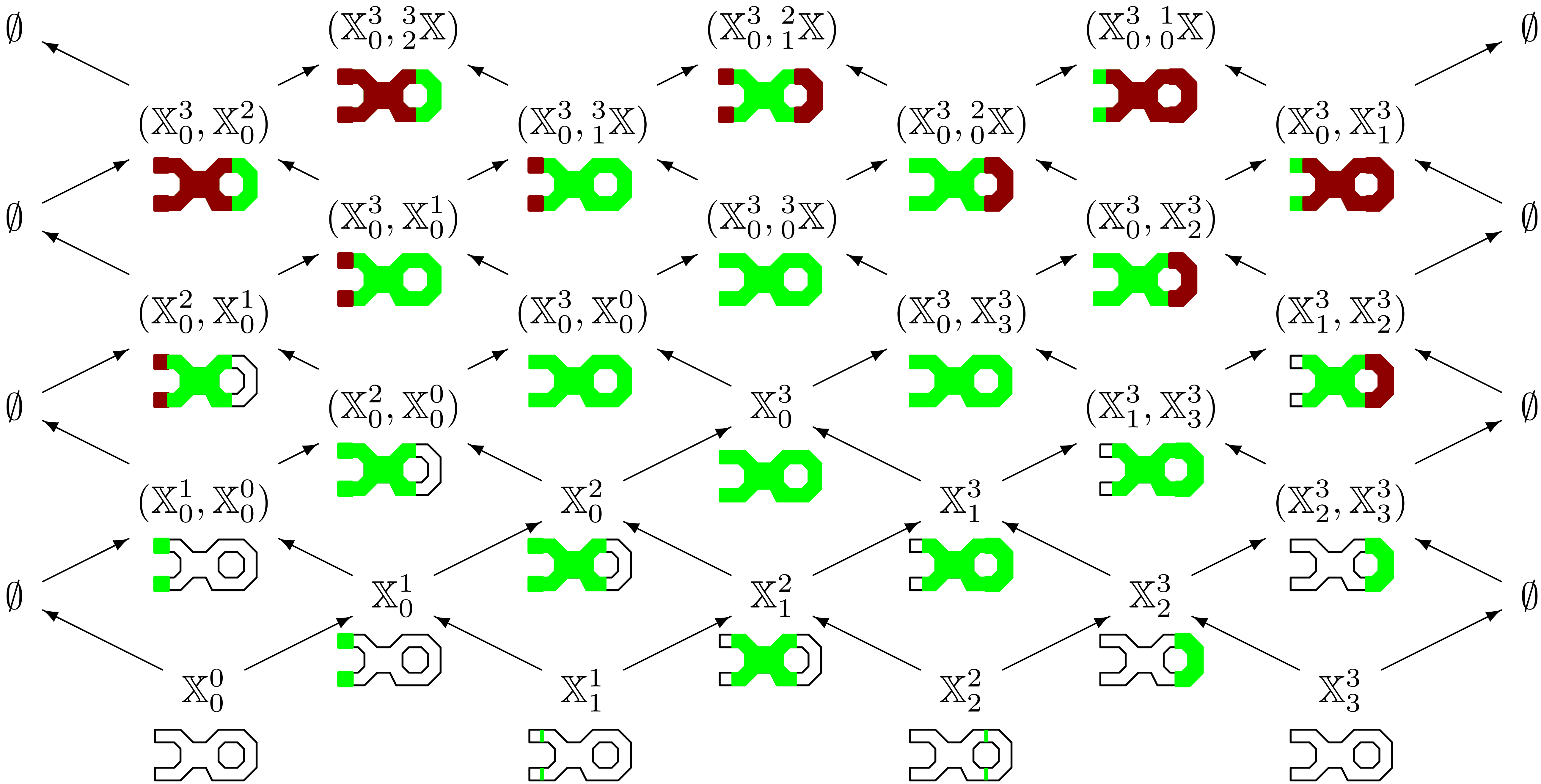


Birth: 

Death: 

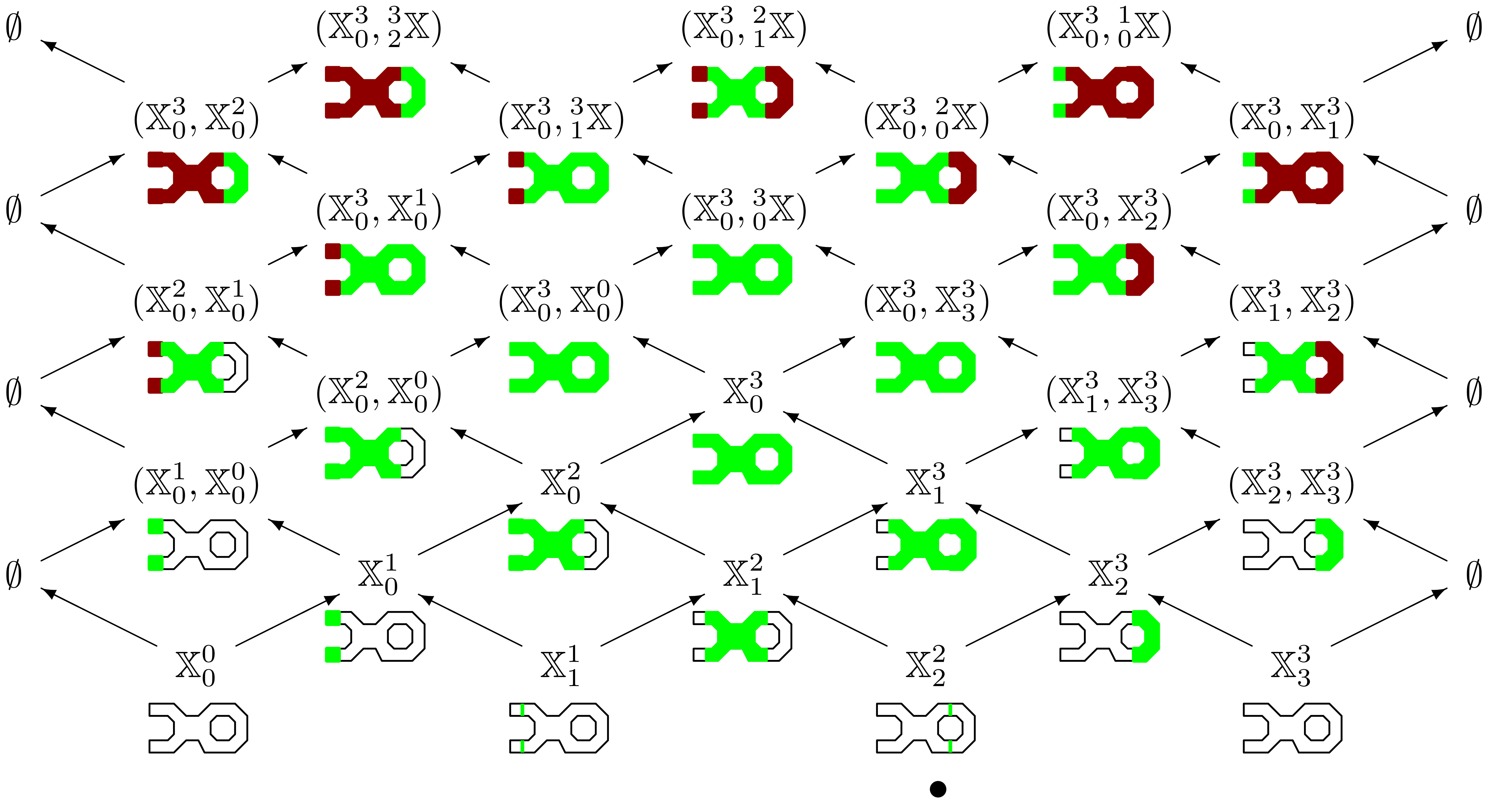
BD: 

Pyramid



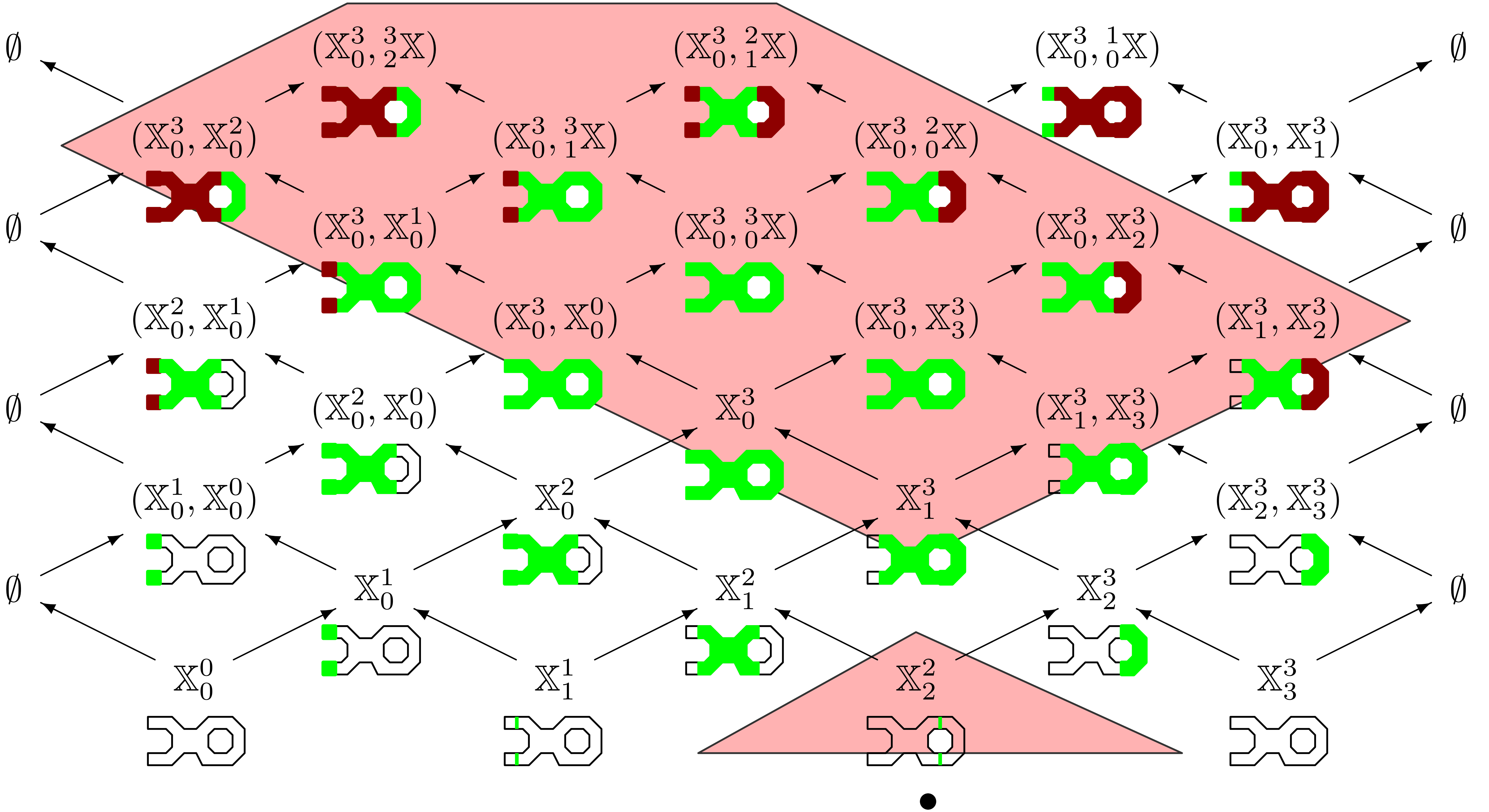
$${}^j_i\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Pyramid



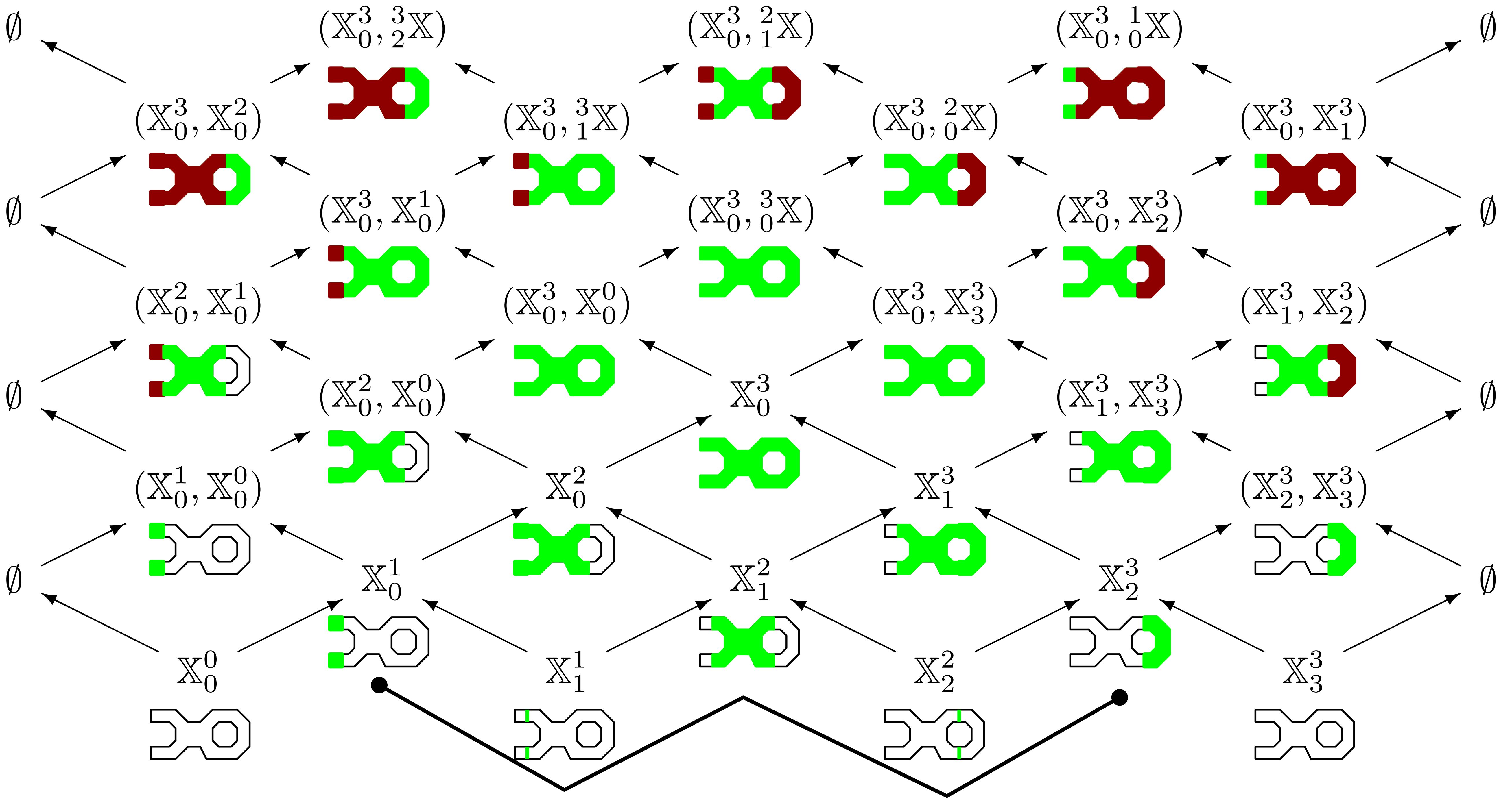
$${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Pyramid



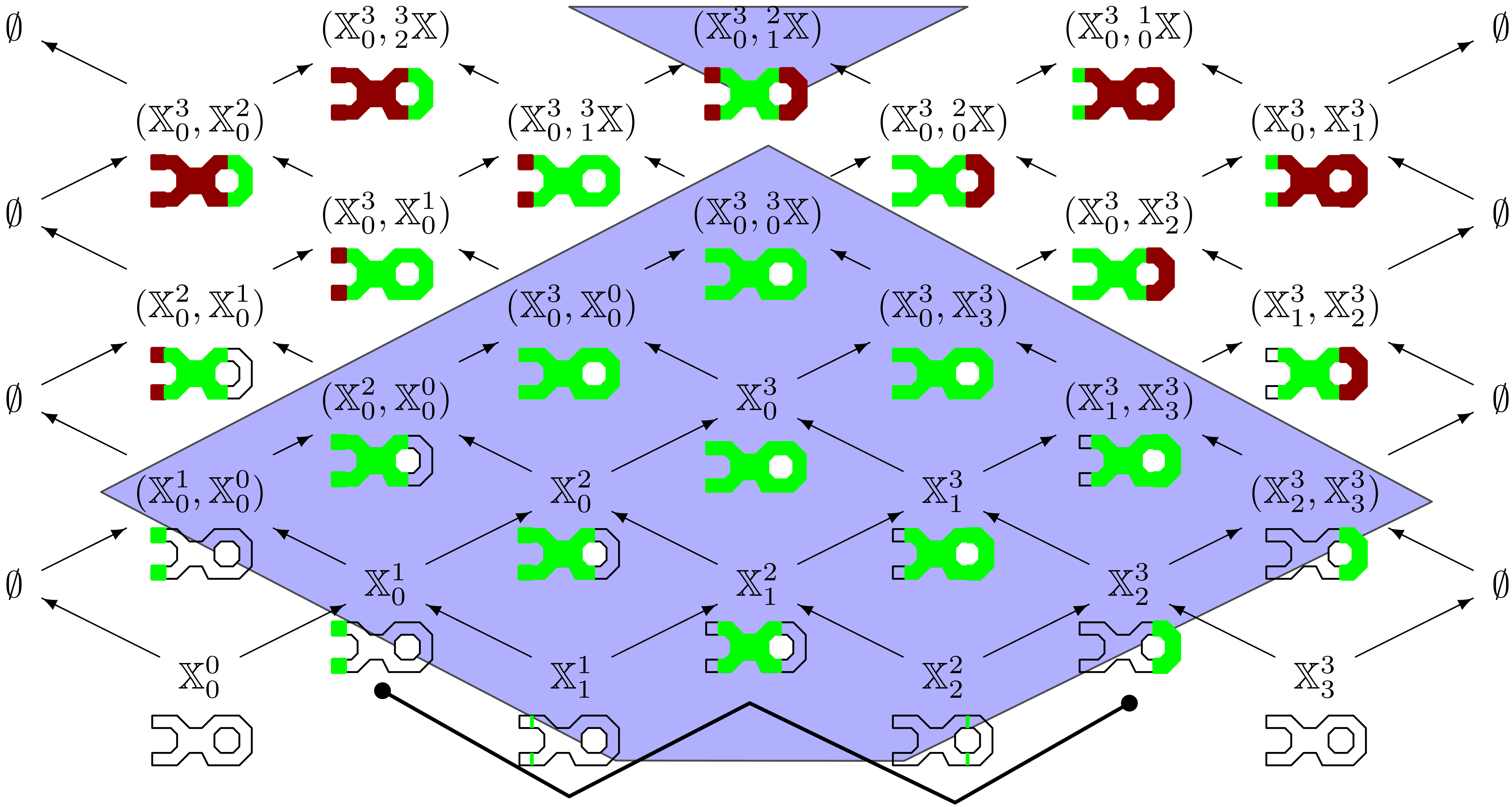
$${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Pyramid



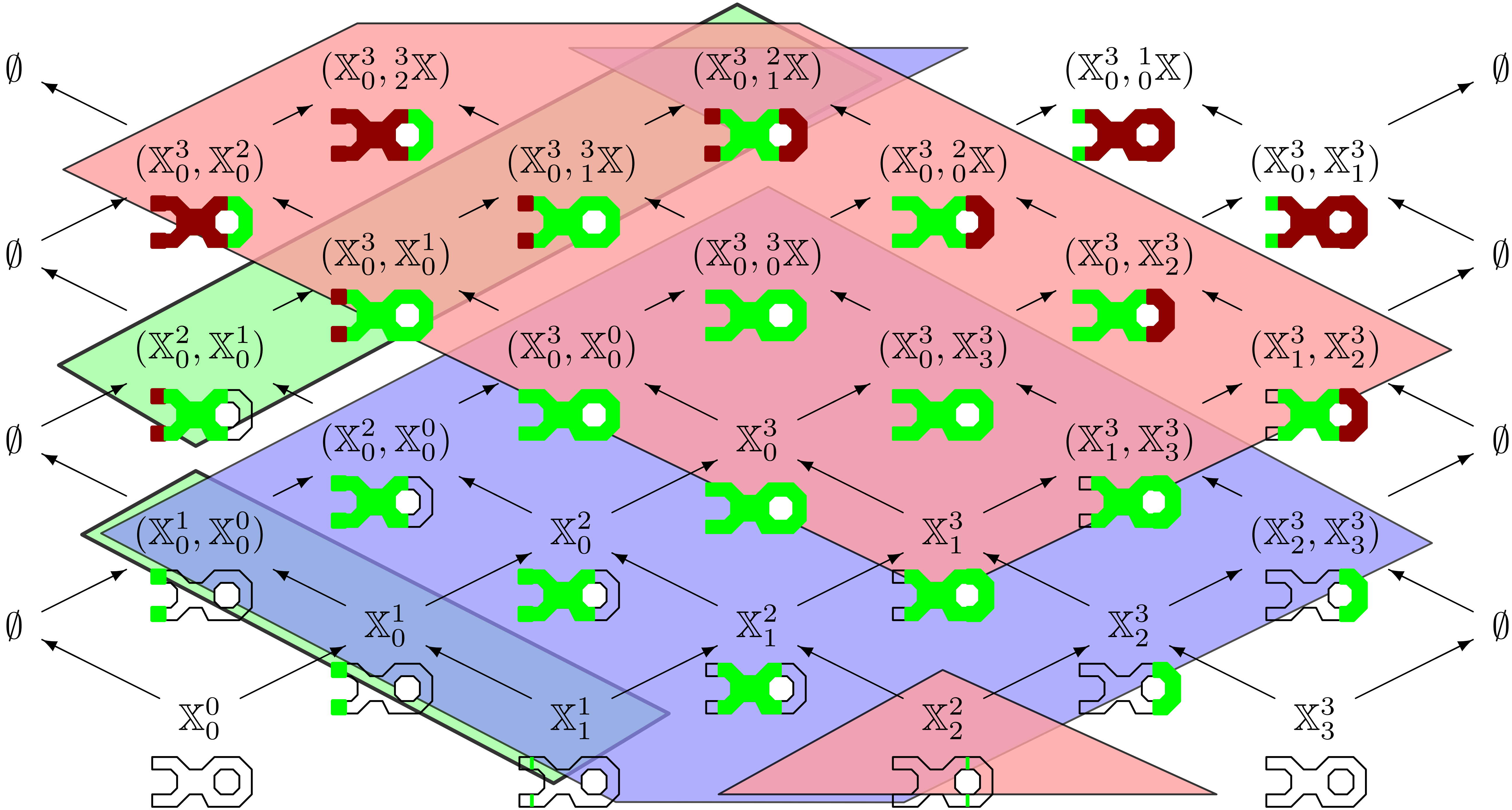
$${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Pyramid



$${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Pyramid



$${}^j_i\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Transformation into Extended Persistence

$$\begin{array}{rccc} T: & (x,y) & \mapsto & (y,x) \\ R: & (x,y) & \mapsto & (-y,-x) \\ 0: & (x,y) & \mapsto & (-x,-y) \end{array}$$

Duality Theorem:

$$\mathsf{Dgm}_r(f) \quad = \quad \mathsf{Dgm}^T_{d-r}(f)$$

Symmetry Theorem:

$$\begin{array}{rcl} \mathrm{Ord}_r(f) & = & \mathrm{Ord}^R_{d-r-1}(-f) \\ \mathrm{Ext}_r(f) & = & \mathrm{Ext}^0_{d-r}(-f) \\ \mathrm{Rel}_r(f) & = & \mathrm{Rel}^R_{d-r+1}(-f) \end{array}$$

Transformation into Extended Persistence

$$\begin{array}{rccc} T: & (x,y) & \mapsto & (y,x) \\ R: & (x,y) & \mapsto & (-y,-x) \\ 0: & (x,y) & \mapsto & (-x,-y) \end{array}$$

Duality Theorem:

$$\mathsf{Dgm}_r(f) = \mathsf{Dgm}^T_{d-r}(f)$$

General

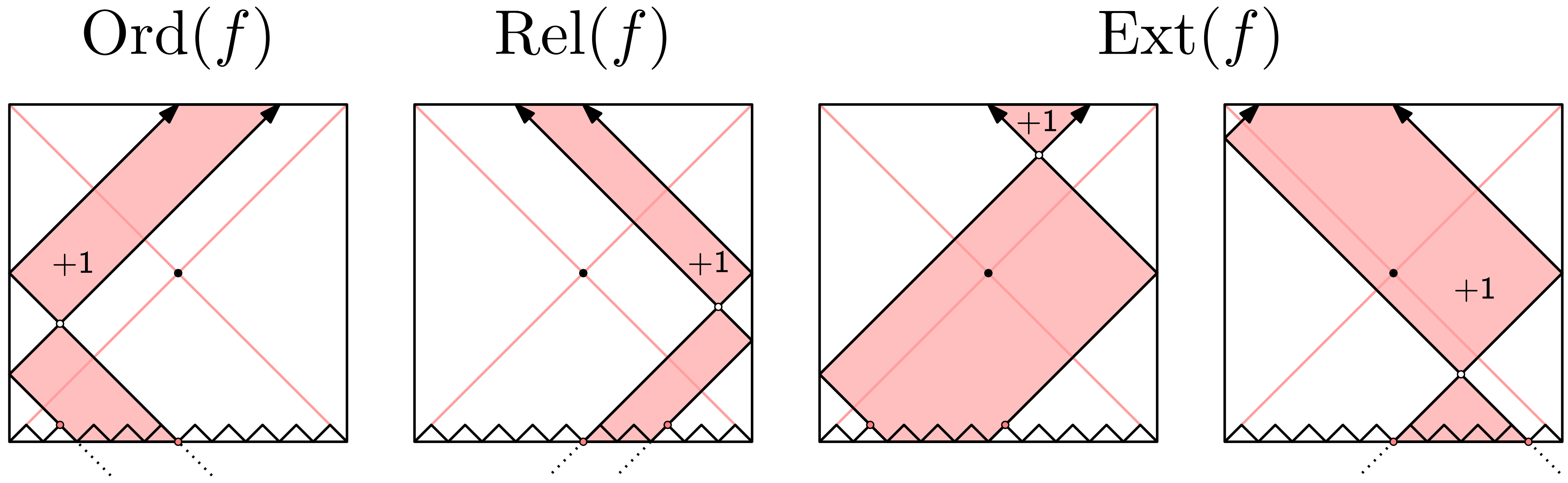
Symmetry Theorem:

$$\begin{array}{rcl} \mathrm{Ord}_r(f) & = & \mathrm{Rel}^0_{r+1}(-f) \\ \mathrm{Ext}_r(f) & = & \mathrm{Ext}^R_r(-f) \\ \mathrm{Rel}_r(f) & = & \mathrm{Ord}^0_{r-1}(-f) \end{array}$$

Manifolds

Symmetry Theorem:

$$\begin{array}{rcl} \mathrm{Ord}_r(f) & = & \mathrm{Ord}^R_{d-r-1}(-f) \\ \mathrm{Ext}_r(f) & = & \mathrm{Ext}^0_{d-r}(-f) \\ \mathrm{Rel}_r(f) & = & \mathrm{Rel}^R_{d-r+1}(-f) \end{array}$$



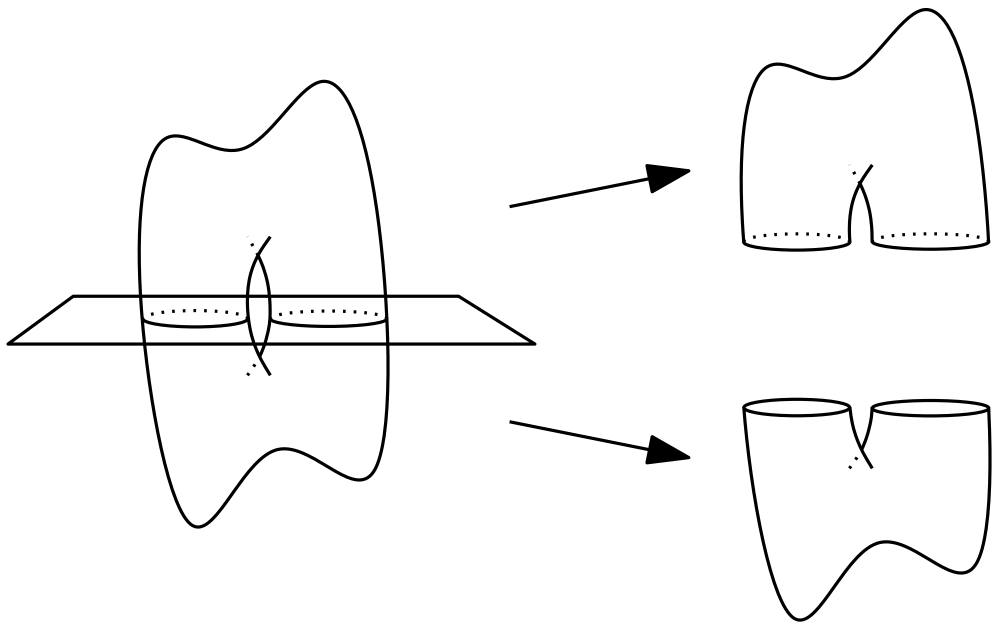
Levelset Zigzag Consequences

- computes persistence using space dependent on the size of the levelset rather than the entire space

Levelset Zigzag Consequences

- computes persistence using space dependent on the size of the levelset rather than the entire space

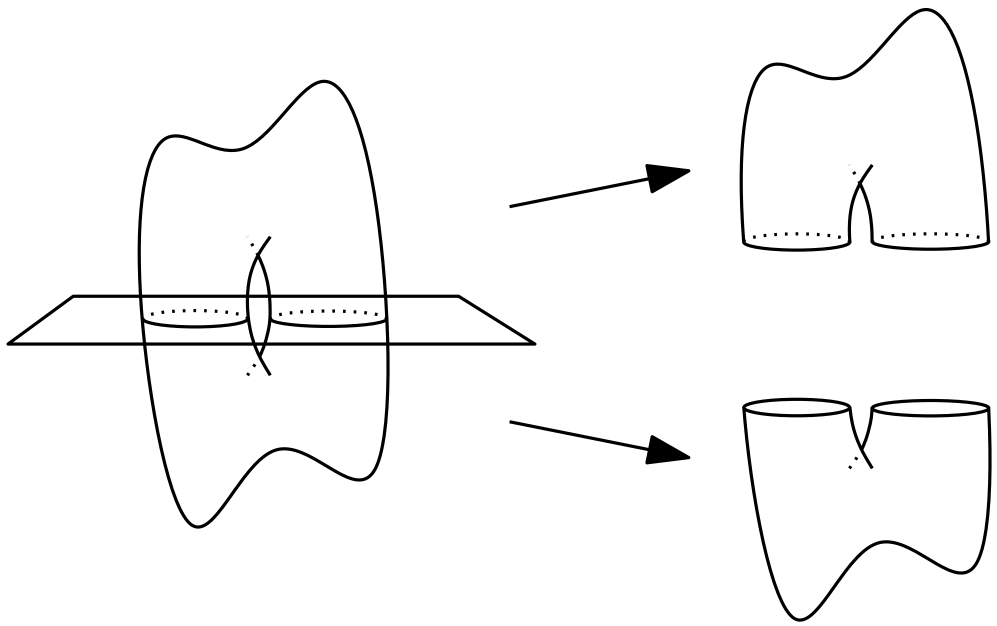
- suggests a parallel algorithm for computing persistence



Levelset Zigzag Consequences

- computes persistence using space dependent on the size of the levelset rather than the entire space

- suggests a parallel algorithm for computing persistence



- resolves an open question relating persistence of f and $-f$

Symmetry Theorem:

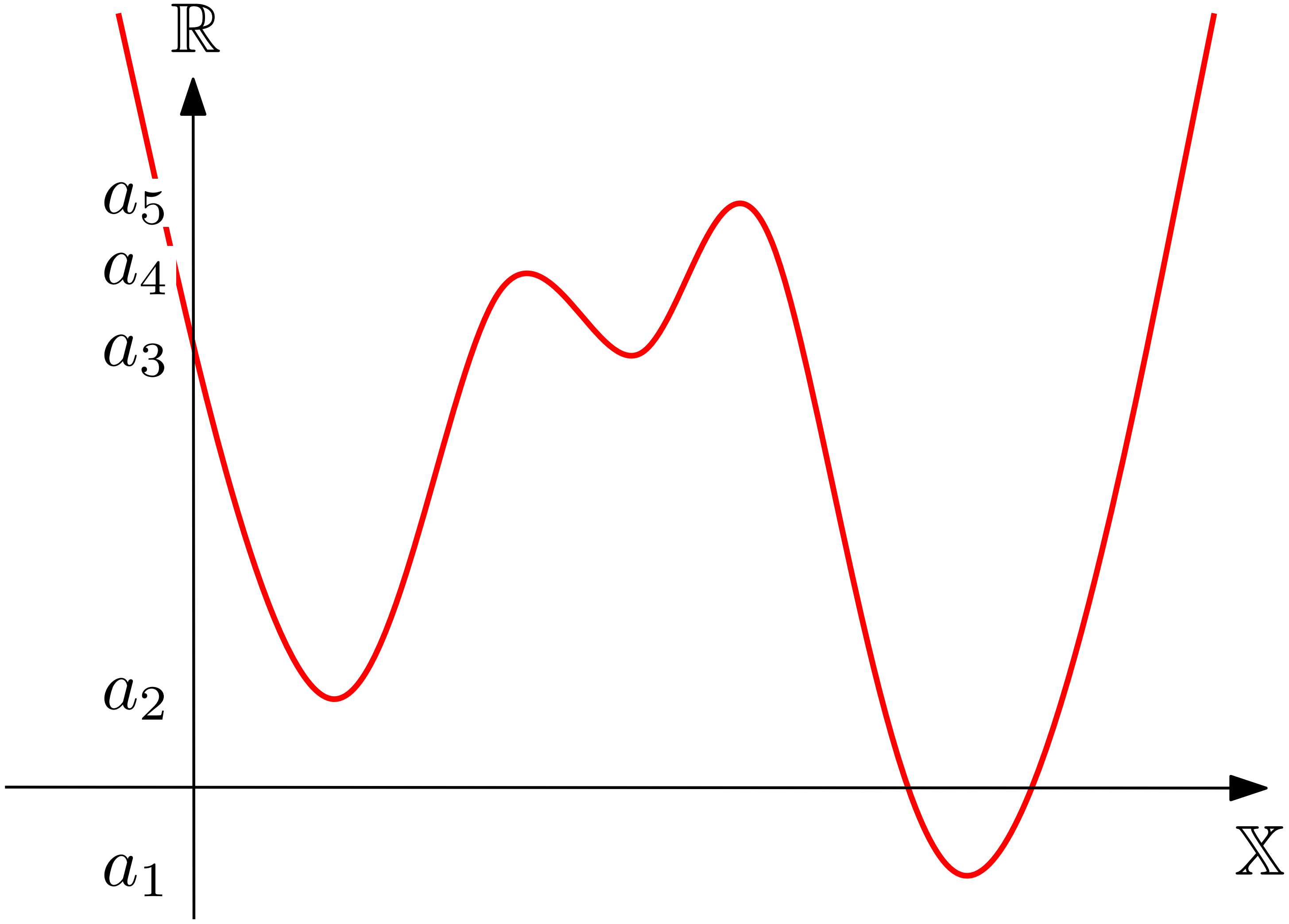
$$\begin{aligned} \mathrm{Ord}_r(f) &= \mathrm{Rel}^0_{r+1}(-f) \\ \mathrm{Ext}_r(f) &= \mathrm{Ext}^R_r(-f) \\ \mathrm{Rel}_r(f) &= \mathrm{Ord}^0_{r-1}(-f) \end{aligned}$$

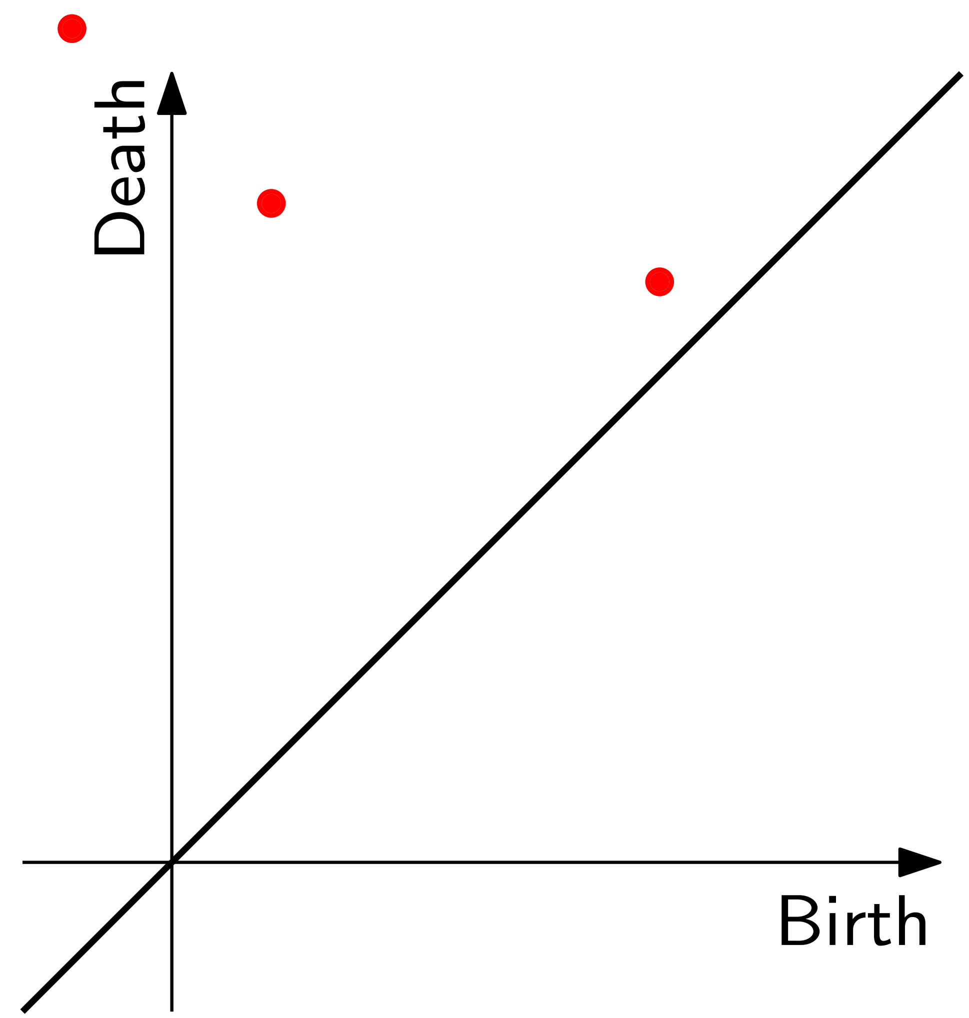
$$\begin{aligned} R: &\quad (x,y) \mapsto (-y,-x) \\ 0: &\quad (x,y) \mapsto (-x,-y) \end{aligned}$$

Stability as a Goal

Stability

Stability Theorem for Persistence Diagrams:
Let $\mathbb{X}$ be a topological space with continuous tame functions $f, g : \mathbb{X} \to \mathbb{R}$. Then the persistence diagrams $\mathrm{Dgm}(f), \mathrm{Dgm}(g)$ satisfy

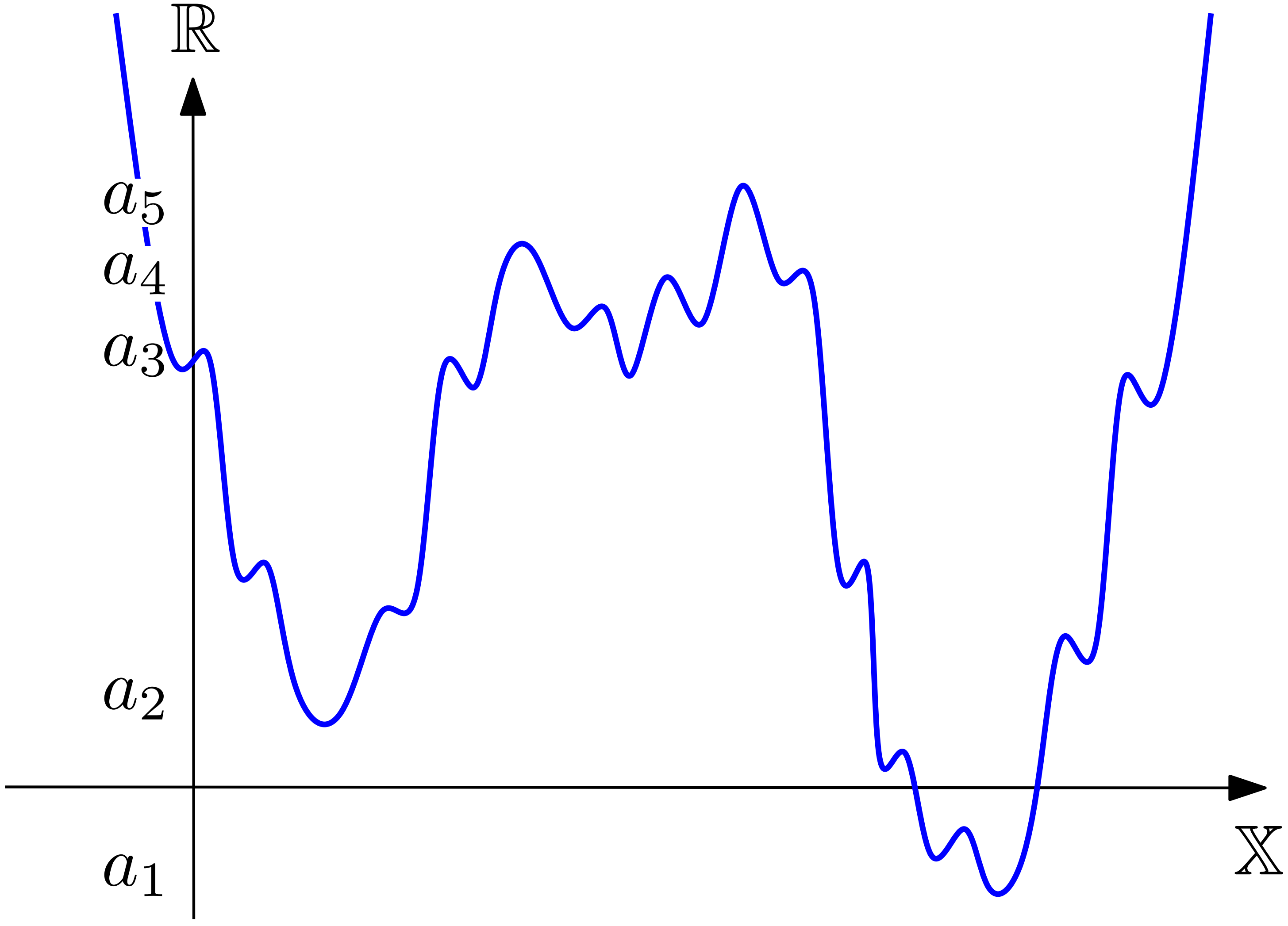


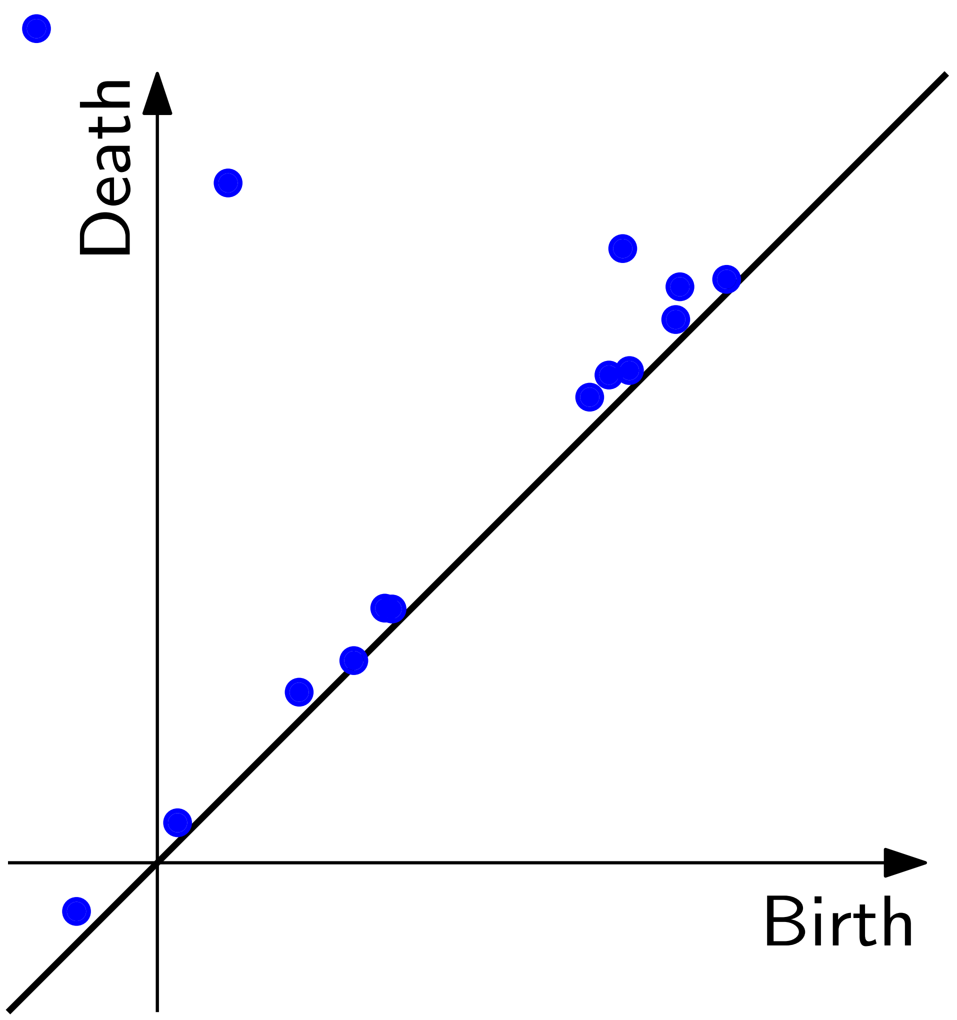


[Cohen-Steiner, Edelsbrunner, Harer '05]

Stability

Stability Theorem for Persistence Diagrams:
Let $\mathbb{X}$ be a topological space with continuous tame functions $f, g : \mathbb{X} \to \mathbb{R}$. Then the persistence diagrams $\mathrm{Dgm}(f), \mathrm{Dgm}(g)$ satisfy



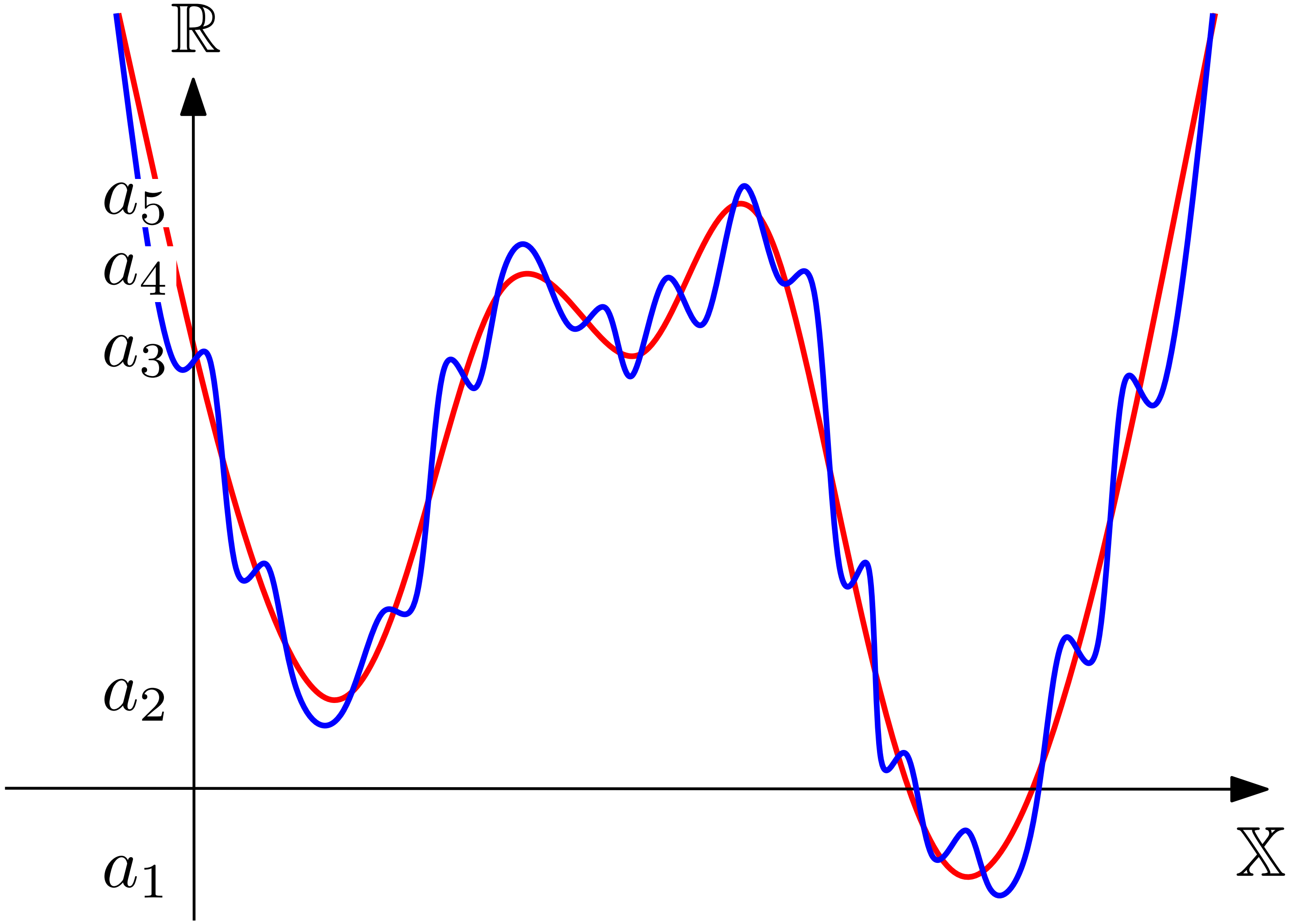


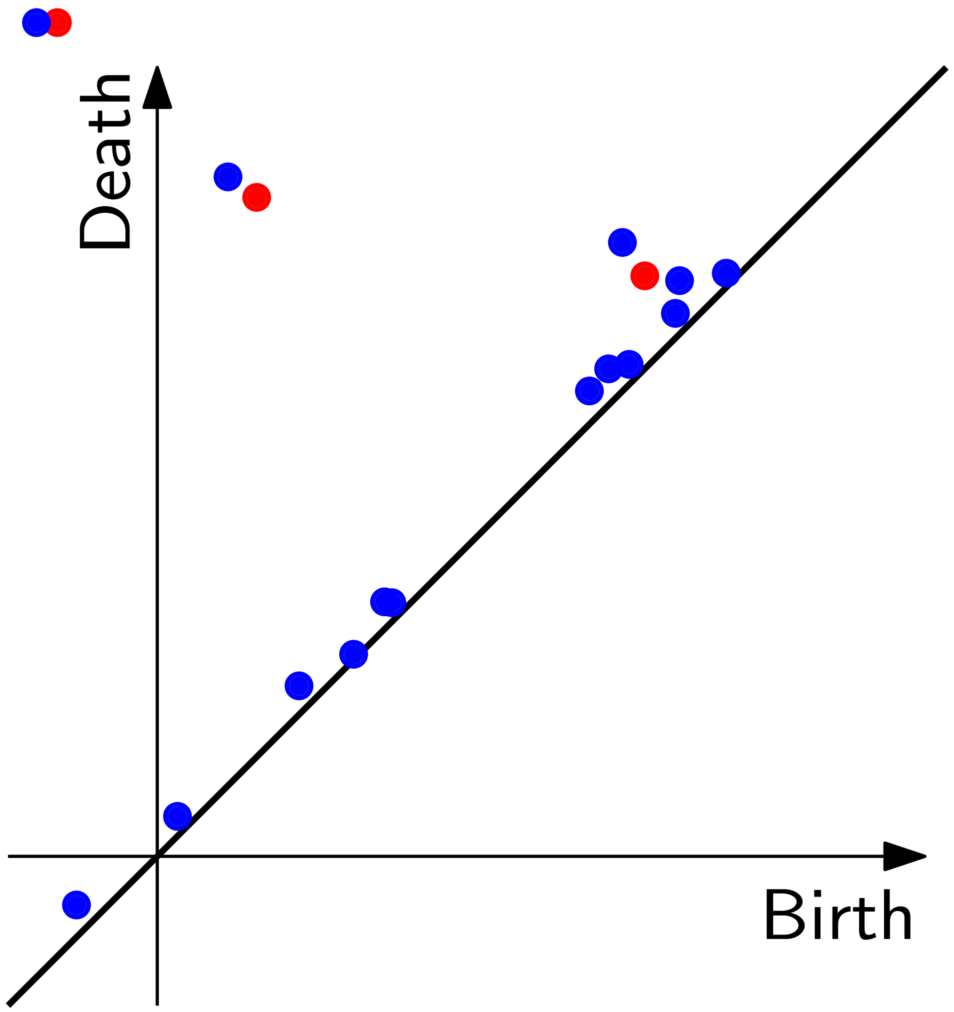
[Cohen-Steiner, Edelsbrunner, Harer '05]

Stability

Stability Theorem for Persistence Diagrams:
Let $\mathbb{X}$ be a topological space with continuous tame functions $f, g : \mathbb{X} \to \mathbb{R}$. Then the persistence diagrams $\mathrm{Dgm}(f), \mathrm{Dgm}(g)$ satisfy

$$W_\infty(\mathrm{Dgm}(f), \mathrm{Dgm}(g)) \leq ||f - g||_\infty$$





[Cohen-Steiner, Edelsbrunner, Harer '05]

Well Groups

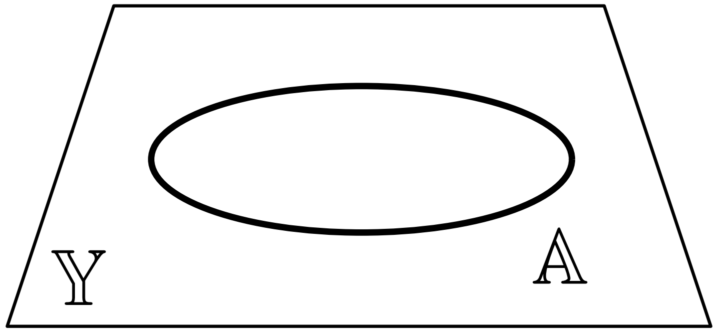
$f : \mathbb{X} \to \mathbb{Y}$, $\mathbb{A} \subseteq \mathbb{Y}$

$\mathbb{X}$ = topological space

$\mathbb{Y}$ = metric space

Stability of $H(f^{-1}(A))$?

$f : \mathbb{X} \to$



[Edelsbrunner, Morozov, Patel]

Well Groups

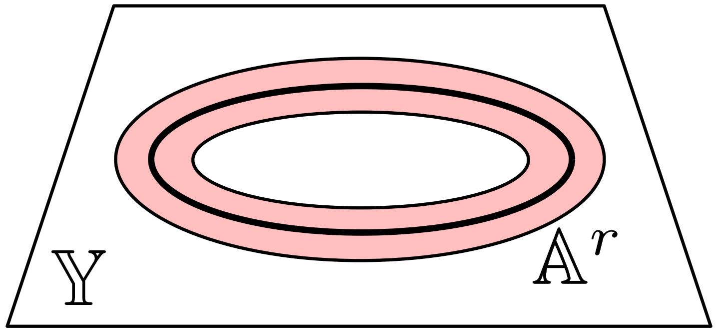
$f : \mathbb{X} \to \mathbb{Y}, \qquad \mathbb{A} \subseteq \mathbb{Y}$

$\mathbb{X}$ = topological space

$\mathbb{Y}$ = metric space

Stability of $H(f^{-1}(A))$?

$f : \mathbb{X} \to$



$$\mathbb{A}^r = \{x : \inf_{a \in \mathbb{A}} \|x - a\| \leq r\}$$

[Edelsbrunner, Morozov, Patel]

Well Groups

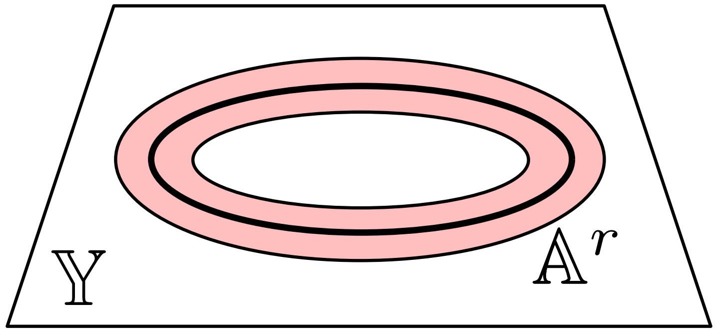
$f : \mathbb{X} \to \mathbb{Y}$, $\mathbb{A} \subseteq \mathbb{Y}$

$\mathbb{X}$ = topological space

$\mathbb{Y}$ = metric space

Stability of $H(f^{-1}(A))$?

$f : \mathbb{X} \to$



$$\mathbb{A}^r = \{x : \inf_{a \in \mathbb{A}} \|x - a\| \leq r\}$$

$h : \mathbb{X} \to \mathbb{Y}$, r-perturbation: $\|h - f\|_\infty \leq r$

[Edelsbrunner, Morozov, Patel]

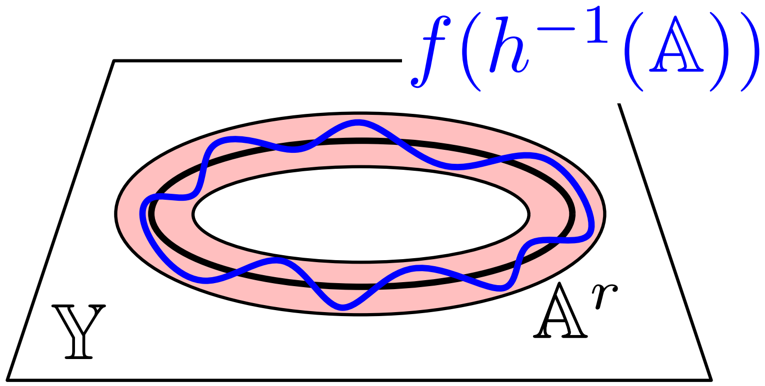
Well Groups

$f : \mathbb{X} \to \mathbb{Y}, \qquad \mathbb{A} \subseteq \mathbb{Y}$

$\mathbb{X}$ = topological space

$\mathbb{Y}$ = metric space

$f : \mathbb{X} \to$



Stability of $H(f^{-1}(A))$?

$$\mathbb{A}^r = \{x : \inf_{a \in \mathbb{A}} \|x - a\| \leq r\}$$

$h : \mathbb{X} \to \mathbb{Y}$, $\qquad r$-perturbation: $\|h - f\|_\infty \leq r$

$h^{-1}(\mathbb{A}) \subseteq f^{-1}(\mathbb{A}^r)$

$\mathsf{j}_h : \mathsf{H}(h^{-1}(\mathbb{A})) \to \mathsf{F}(r)$

$\mathsf{F}(r) = \mathsf{H}(f^{-1}(\mathbb{A}^r))$

[Edelsbrunner, Morozov, Patel]

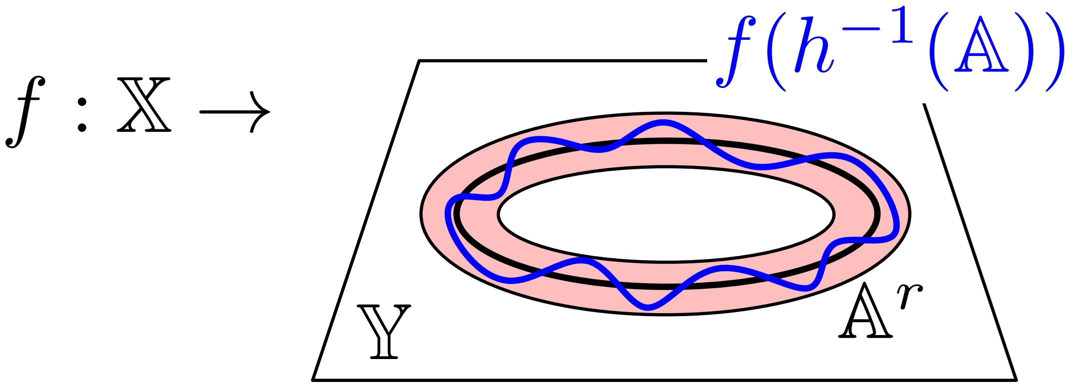
Well Groups

$f : \mathbb{X} \to \mathbb{Y}, \qquad \mathbb{A} \subseteq \mathbb{Y}$

$\mathbb{X}$ = topological space

$\mathbb{Y}$ = metric space

Stability of $H(f^{-1}(A))$?



$$\mathbb{A}^r = \{x : \inf_{a \in \mathbb{A}} \|x - a\| \leq r\}$$

$h : \mathbb{X} \to \mathbb{Y}, \qquad r$-perturbation: $\|h - f\|_\infty \leq r$

$h^{-1}(\mathbb{A}) \subseteq f^{-1}(\mathbb{A}^r)$

$\mathsf{j}_h : \mathsf{H}(h^{-1}(\mathbb{A})) \to \mathsf{F}(r)$

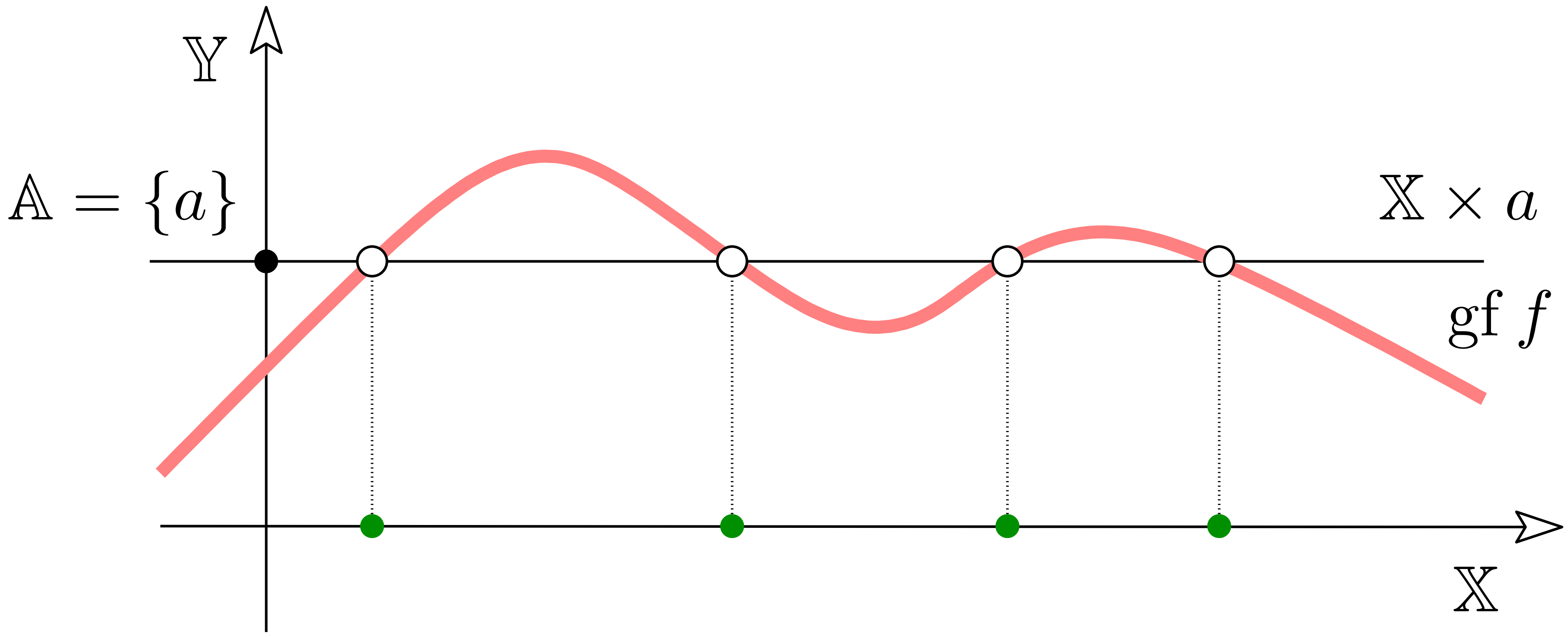
$\mathsf{F}(r) = \mathsf{H}(f^{-1}(\mathbb{A}^r))$

Definition: The **well group** $\mathsf{U}(r) \subseteq \mathsf{F}(r)$:

$$\mathsf{U}(r) = \bigcap_{\|h-f\|_\infty \leq r} \mathrm{im}\,\mathsf{j}_h.$$

[Edelsbrunner, Morozov, Patel]

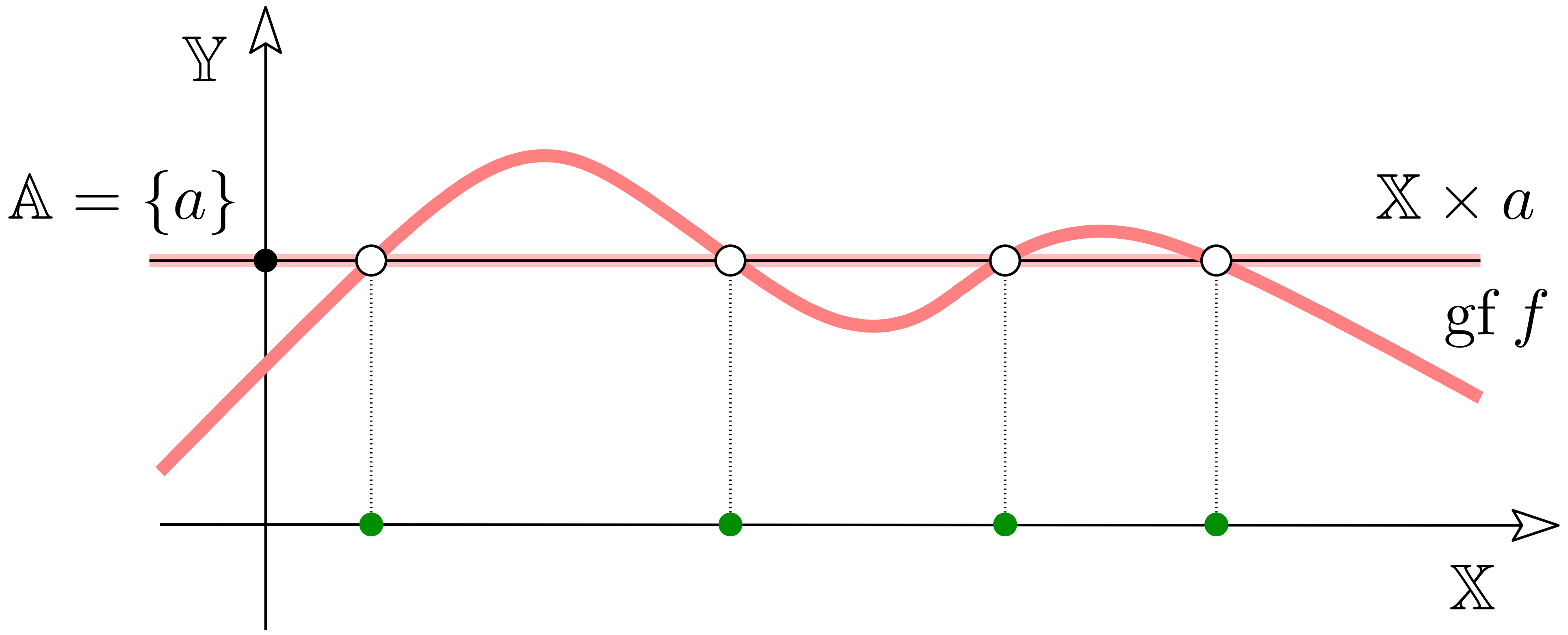
Well Diagram Example



$\mathsf{F}(r)$
$\mathsf{U}(r)$

$\mathsf{F}(r) = \mathsf{H}(f^{-1}(\mathbb{A}^r))$
$\mathsf{U}(r) =$ well group

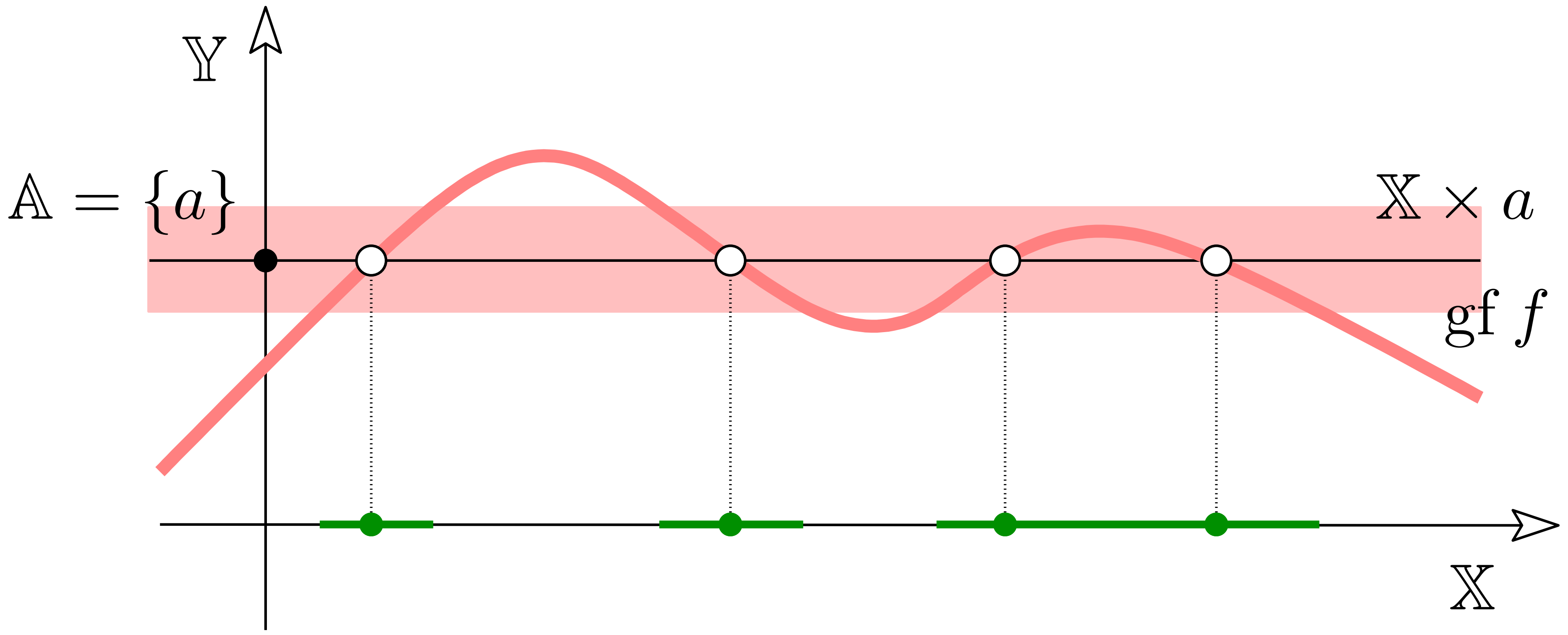
Well Diagram Example



	$[0, r_3)$
F(r)	4
U(r)	4

F(r) = H$(f^{-1}(\mathbb{A}^r))$
U(r) = well group

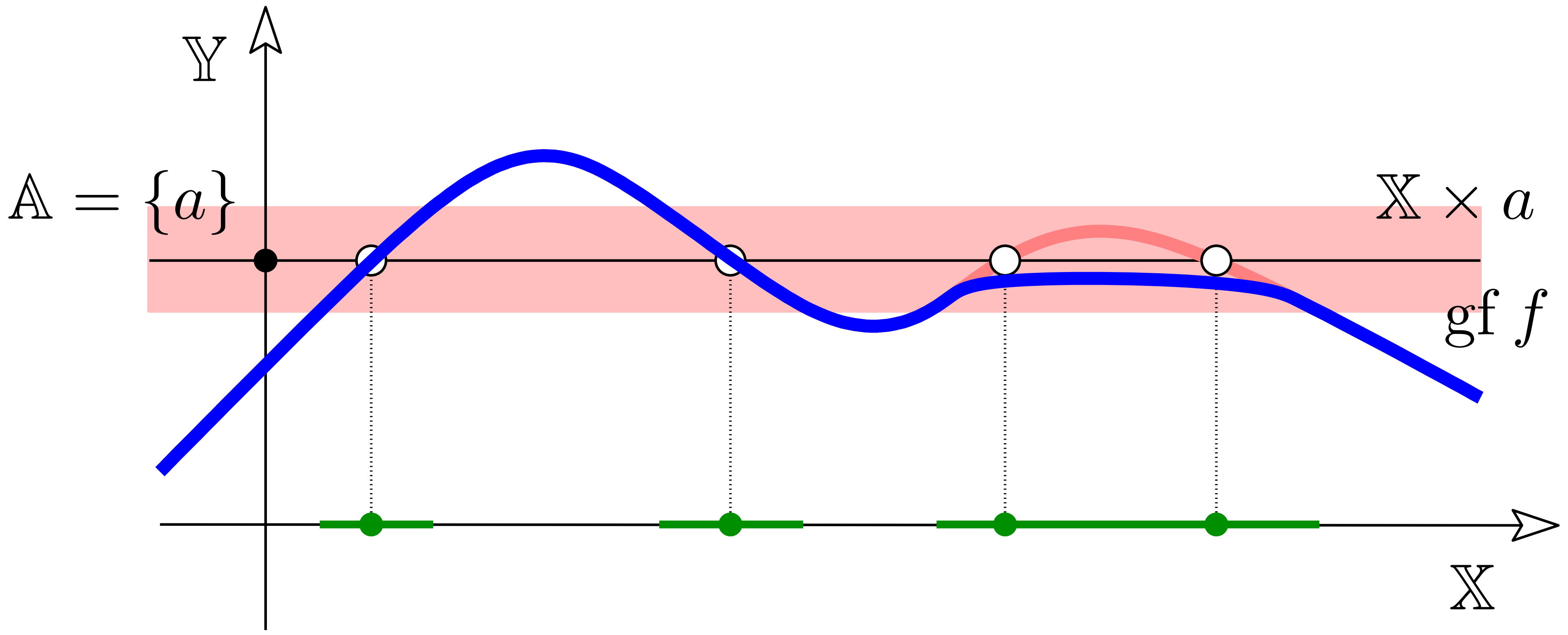
Well Diagram Example



	$[0, r_3)$	$[r_3, r_2)$
F(r)	4	3
U(r)	4	2

F$(r) =$ H$(f^{-1}(\mathbb{A}^r))$
U$(r) =$ well group

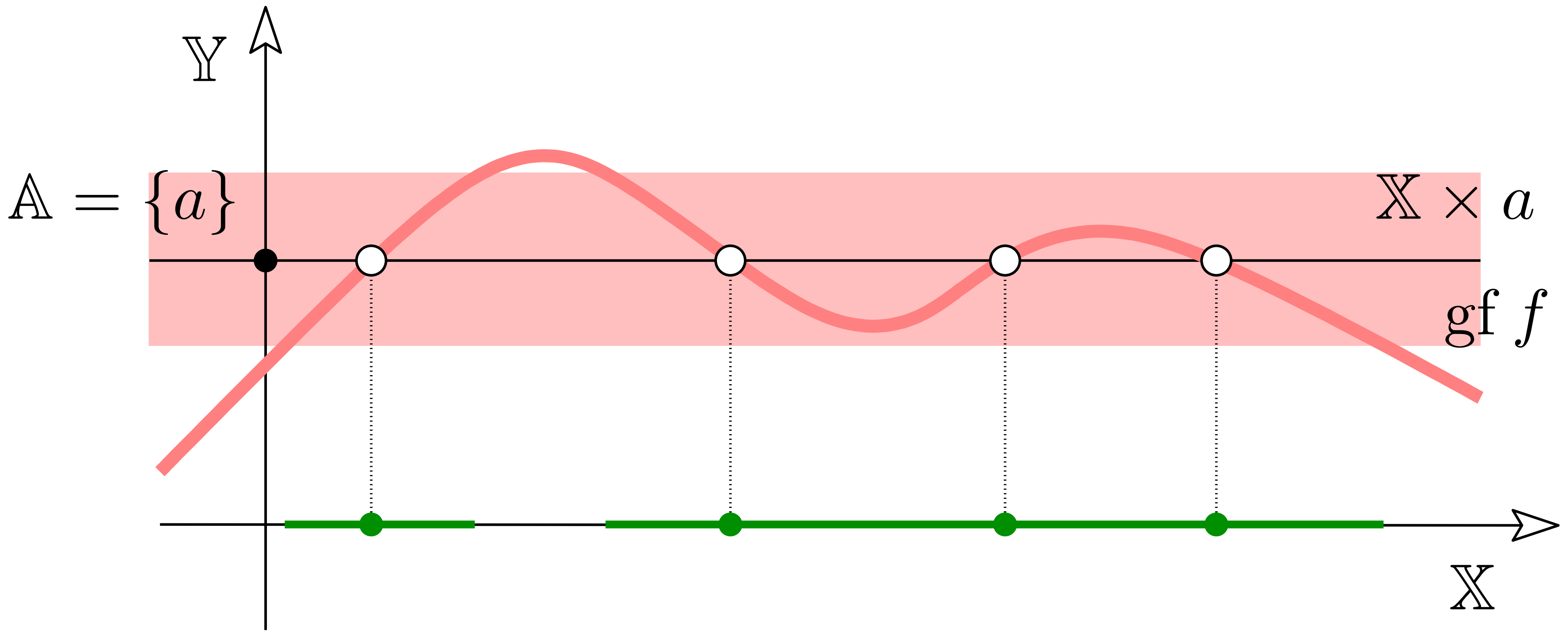
Well Diagram Example



	$[0, r_3)$	$[r_3, r_2)$
F(r)	4	3
U(r)	4	2

F$(r) =$ H$(f^{-1}(\mathbb{A}^r))$
U$(r) =$ well group

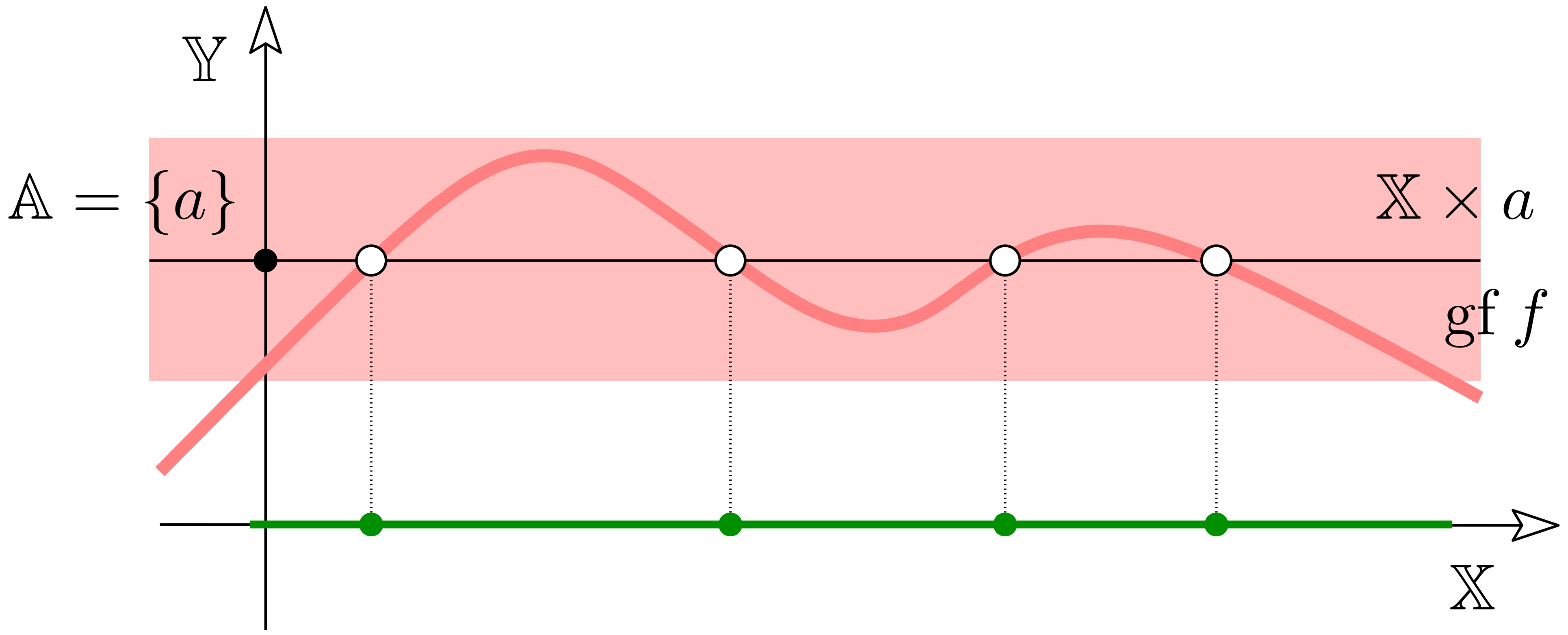
Well Diagram Example



	$[0, r_3)$	$[r_3, r_2)$	$[r_2, r_1)$
$\mathsf{F}(r)$	4	3	2
$\mathsf{U}(r)$	4	2	2

$\mathsf{F}(r) = \mathsf{H}(f^{-1}(\mathbb{A}^r))$
$\mathsf{U}(r) =$ well group

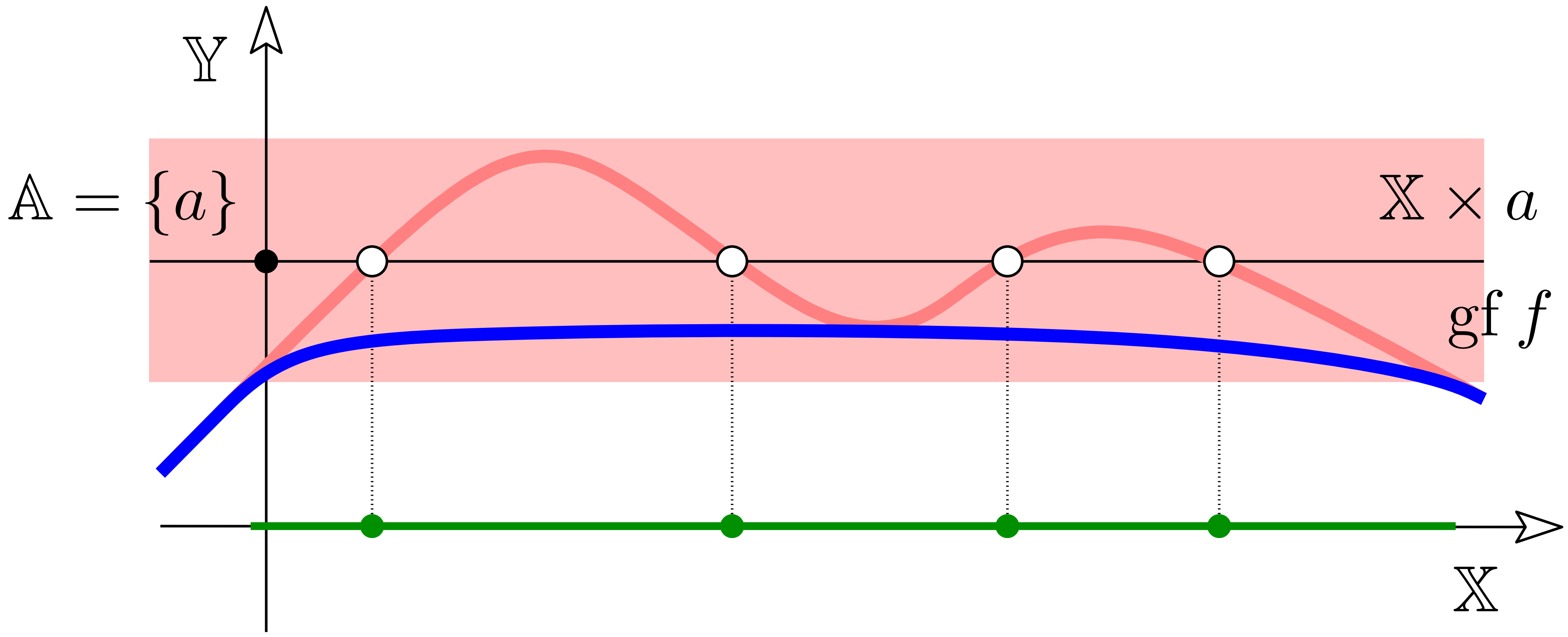
Well Diagram Example



	$[0, r_3)$	$[r_3, r_2)$	$[r_2, r_1)$	$[r_1, \infty)$
F(r)	4	3	2	1
U(r)	4	2	2	0

F(r) = H$(f^{-1}(\mathbb{A}^r))$
U(r) = well group

Well Diagram Example

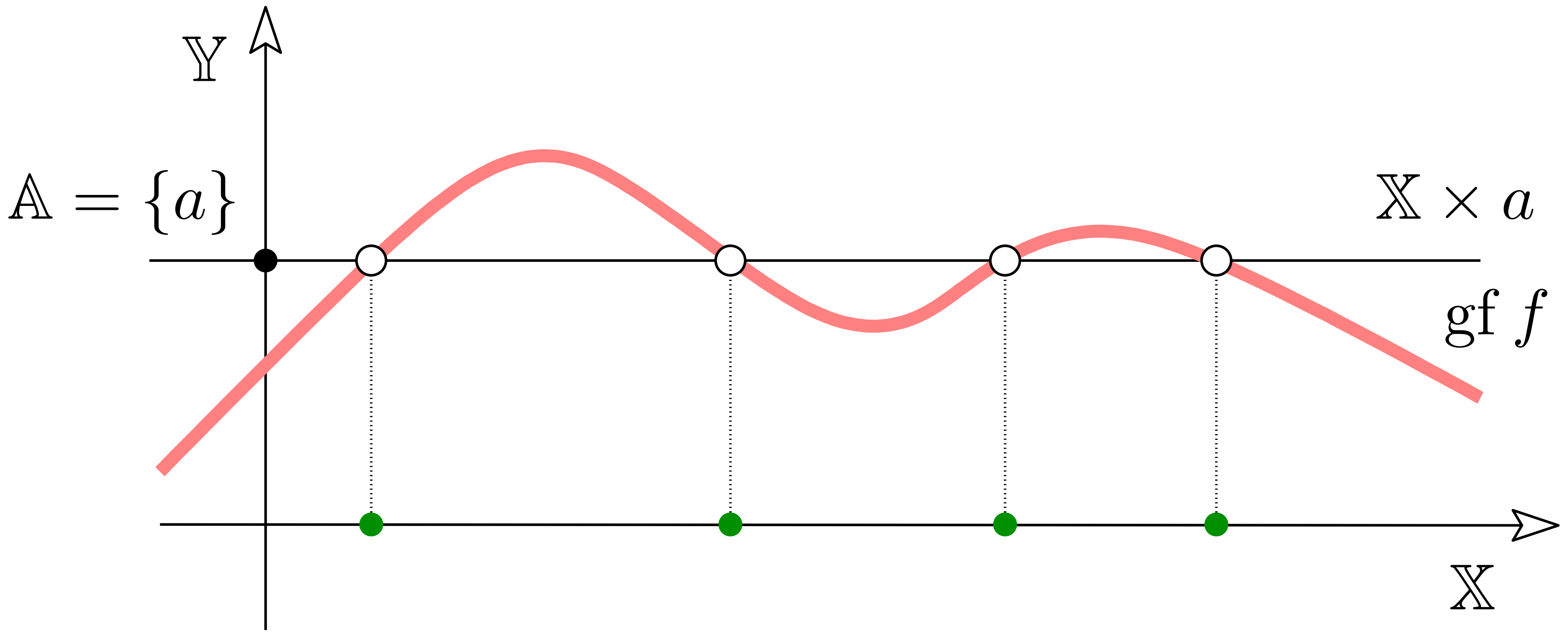


	$[0, r_3)$	$[r_3, r_2)$	$[r_2, r_1)$	$[r_1, \infty)$
F(r)	4	3	2	1
U(r)	4	2	2	0

F(r) = H$(f^{-1}(\mathbb{A}^r))$
U(r) = well group

Well Diagram Example

Well diagram $\mathrm{Dgm}(f, \mathbb{A}) = \{r_1, r_1, r_3, r_3, \infty \times 0\}$

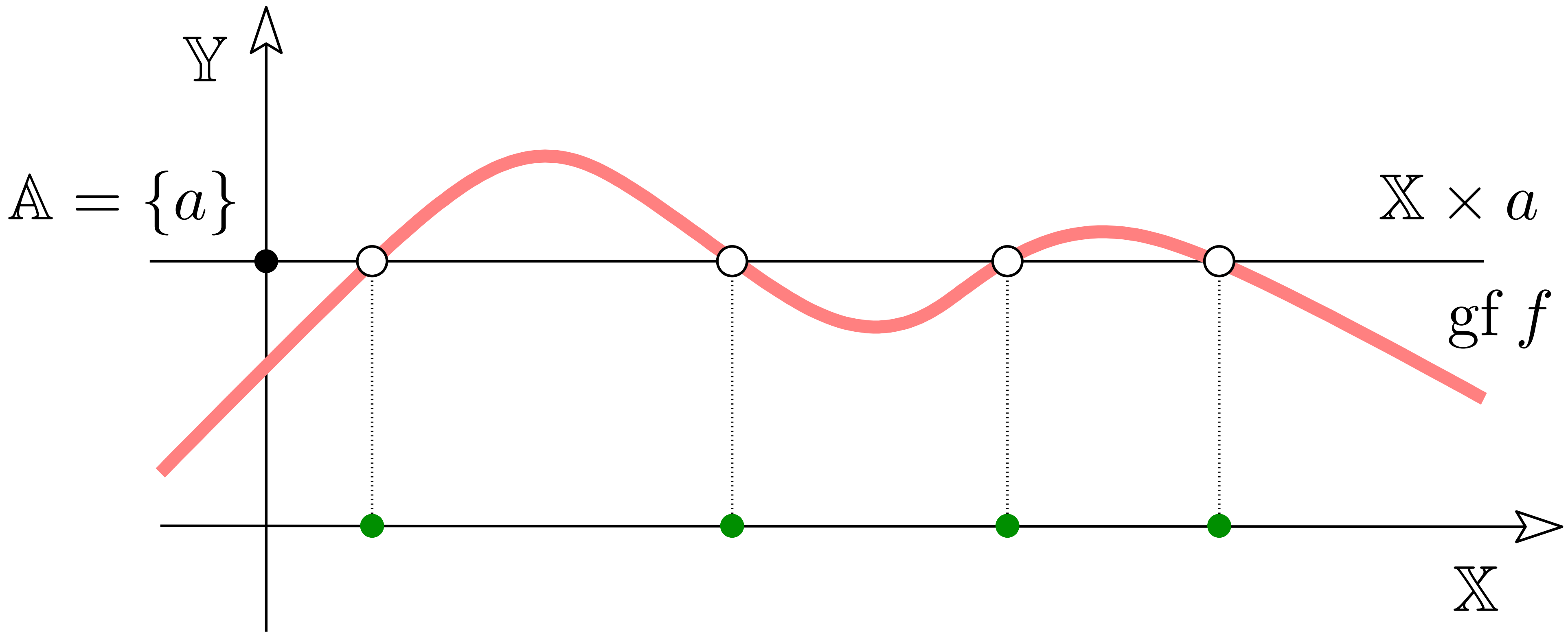


	$[0, r_3)$	$[r_3, r_2)$	$[r_2, r_1)$	$[r_1, \infty)$
F(r)	4	3	2	1
U(r)	4	2	2	0

F$(r) =$ H$(f^{-1}(\mathbb{A}^r))$
U$(r) =$ well group

Well Diagram Example

Well diagram $\text{Dgm}(f, \mathbb{A}) = \{r_1, r_1, r_3, r_3, \infty \times 0\}$



Stability Theorem for Well Diagrams: Let $\text{Dgm}(f, \mathbb{A})$ and $\text{Dgm}(g, \mathbb{A})$ be the well diagrams of the admissible maps $f, g : \mathbb{X} \to \mathbb{Y}$, where $\mathbb{X}, \mathbb{Y}$, and $\mathbb{A} \subseteq \mathbb{Y}$. Then

$$W_\infty(\text{Dgm}(f, \mathbb{A}), \text{Dgm}(g, \mathbb{A})) \leq \|f - g\|_\infty.$$

Fixed Points

$b : \mathbb{R}^m \to \mathbb{R}^m$
$f(x) = x - b(x)$

Fixed point: $b(x) = x$
$f(x) = 0$

$\mathbb{X} = \mathbb{Y} = \mathbb{R}^m \qquad \mathbb{A} = \{0\}$

Fixed Points

$b : \mathbb{R}^m \to \mathbb{R}^m$
$f(x) = x - b(x)$

Fixed point: $b(x) = x$
$f(x) = 0$

$\mathbb{X} = \mathbb{Y} = \mathbb{R}^m \qquad \mathbb{A} = \{0\}$

Brouwer's Fixed Point Theorem: Every continuous mapping $b : \mathbb{B}^m \to \mathbb{B}^m$ has a fixed point.

Fixed Points

$b : \mathbb{R}^m \to \mathbb{R}^m$

$f(x) = x - b(x)$

Fixed point: $b(x) = x$

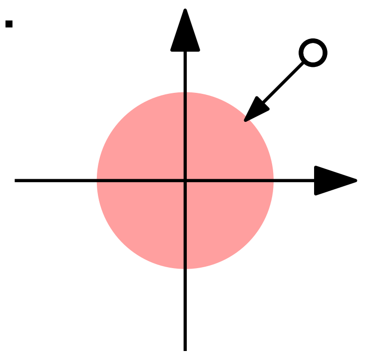
$f(x) = 0$

$\mathbb{X} = \mathbb{Y} = \mathbb{R}^m \qquad \mathbb{A} = \{0\}$

Brouwer's Fixed Point Theorem: Every continuous mapping $b : \mathbb{B}^m \to \mathbb{B}^m$ has a fixed point.

Proof:

Extend b to $\mathbb{R}^m \to \mathbb{R}^m$: $b(x) = b(x/\|x\|_2)$ if $x \notin \mathbb{B}^m$.



Fixed Points

$b : \mathbb{R}^m \to \mathbb{R}^m$
$f(x) = x - b(x)$

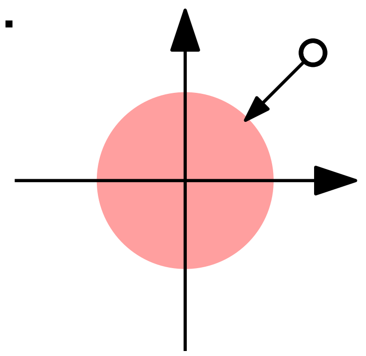
Fixed point: $b(x) = x$
$f(x) = 0$

$\mathbb{X} = \mathbb{Y} = \mathbb{R}^m \qquad \mathbb{A} = \{0\}$

Brouwer's Fixed Point Theorem: Every continuous mapping $b : \mathbb{B}^m \to \mathbb{B}^m$ has a fixed point.

Proof:

Extend b to $\mathbb{R}^m \to \mathbb{R}^m$: $b(x) = b(x/\|x\|_2)$ if $x \notin \mathbb{B}^m$.



$f(x) = x - b(x) \qquad g(x) = x$

Fixed Points

$b : \mathbb{R}^m \to \mathbb{R}^m$
$f(x) = x - b(x)$

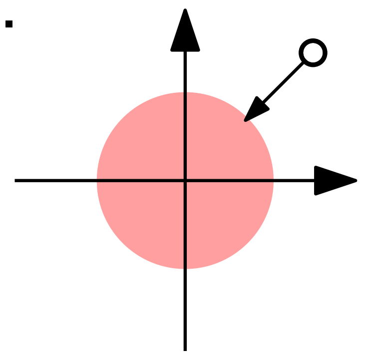
Fixed point: $b(x) = x$
$f(x) = 0$

$\mathbb{X} = \mathbb{Y} = \mathbb{R}^m \qquad \mathbb{A} = \{0\}$

Brouwer's Fixed Point Theorem: Every continuous mapping $b : \mathbb{B}^m \to \mathbb{B}^m$ has a fixed point.

Proof:

Extend b to $\mathbb{R}^m \to \mathbb{R}^m$: $b(x) = b(x/\|x\|_2)$ if $x \notin \mathbb{B}^m$.



$f(x) = x - b(x) \qquad g(x) = x$

$$\begin{aligned} \|f - g\|_\infty &= \sup_{x \in \mathbb{R}^m} \|f(x) - g(x)\|_2 \\ &= \sup_{x \in \mathbb{R}^m} \|b(x)\|_2 = 1 \end{aligned}$$

Dgm$(g, \mathbb{A} = \{0\})$ has a point at infinity $\Rightarrow$ so does the diagram of f.

Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathsf{im}(\mathsf{H}(f^{-1}(a - r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

$$\mathsf{B}_{+r} = \mathsf{im}(\mathsf{H}(f^{-1}(a + r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathsf{im}(\mathsf{H}(f^{-1}(a-r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$
$$\mathsf{B}_{+r} = \mathsf{im}(\mathsf{H}(f^{-1}(a+r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:
$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$

Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a-r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$
$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a+r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:

$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$

Proof sketch:

Show $\mathsf{B}_{-r} \cap \mathsf{B}_{+r} \subseteq \mathsf{U}$.

Let $\alpha \in \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$.

Let h be an r-perturbation.

$$\left.\begin{array}{l} \mathbb{C} = \{x \in \mathbb{A}^r \mid h(x) \leq a\} \\ \mathbb{D} = \{x \in \mathbb{A}^r \mid h(x) \geq a\} \end{array}\right\} \text{support } \alpha$$

$$\mathsf{H}(h^{-1}(a))) \to \mathsf{H}(\mathbb{C}) \oplus \mathsf{H}(\mathbb{D}) \to \mathsf{H}(\mathbb{A}^r)$$

By M-V, α is supported by $h^{-1}(a)$.

Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

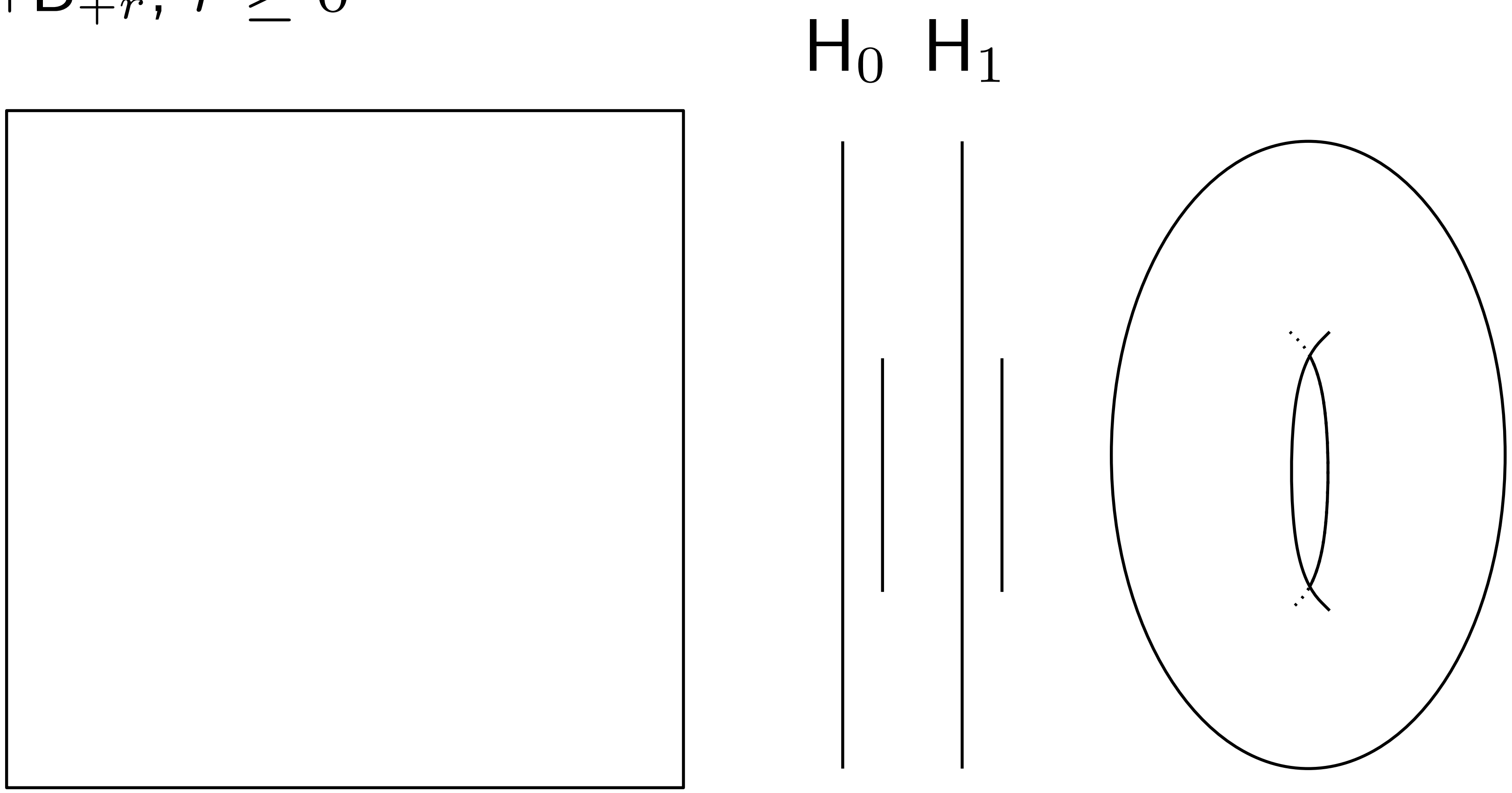
One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a - r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a + r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:

$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$



Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

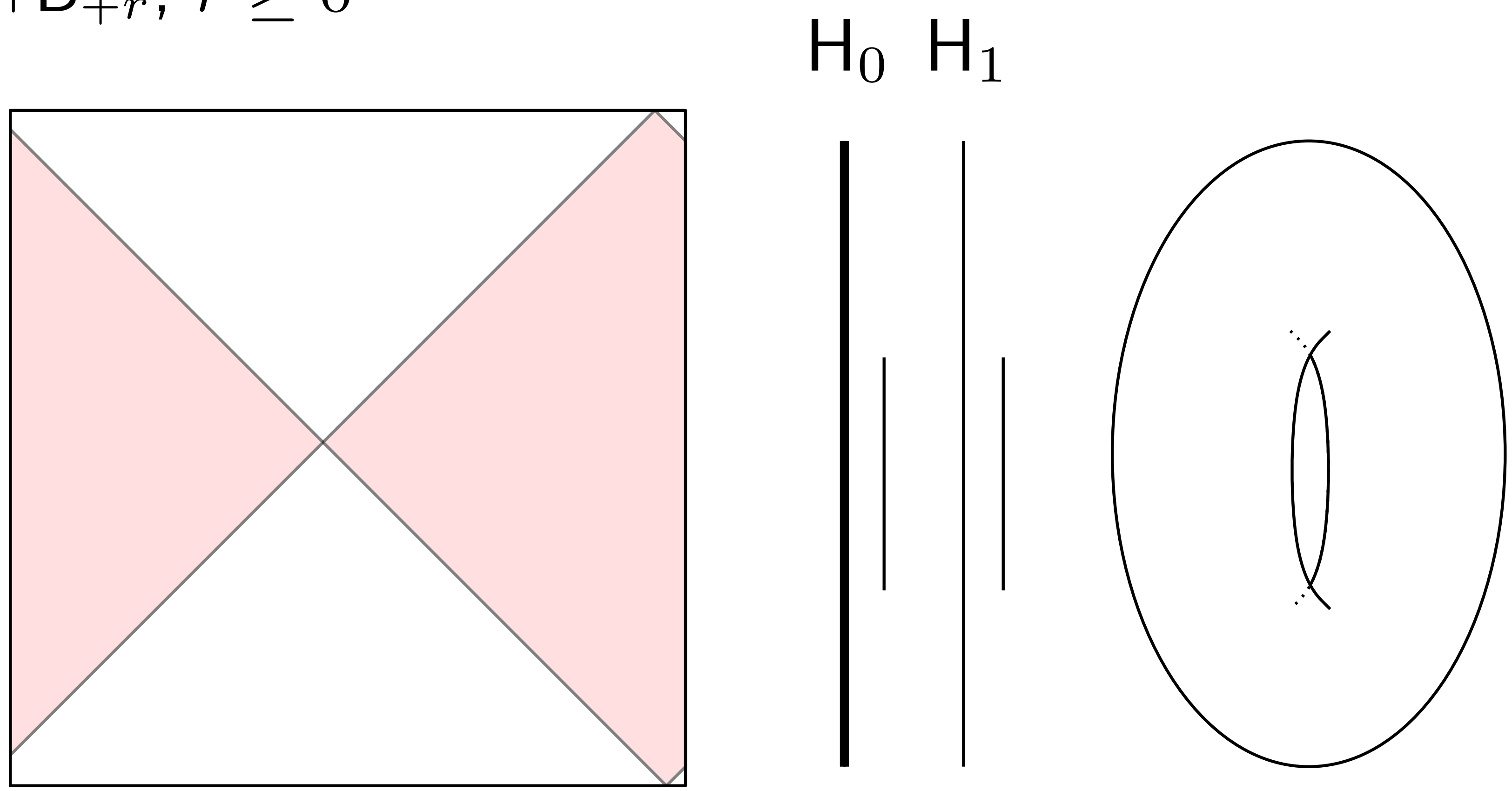
One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a - r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a + r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:

$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$



Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

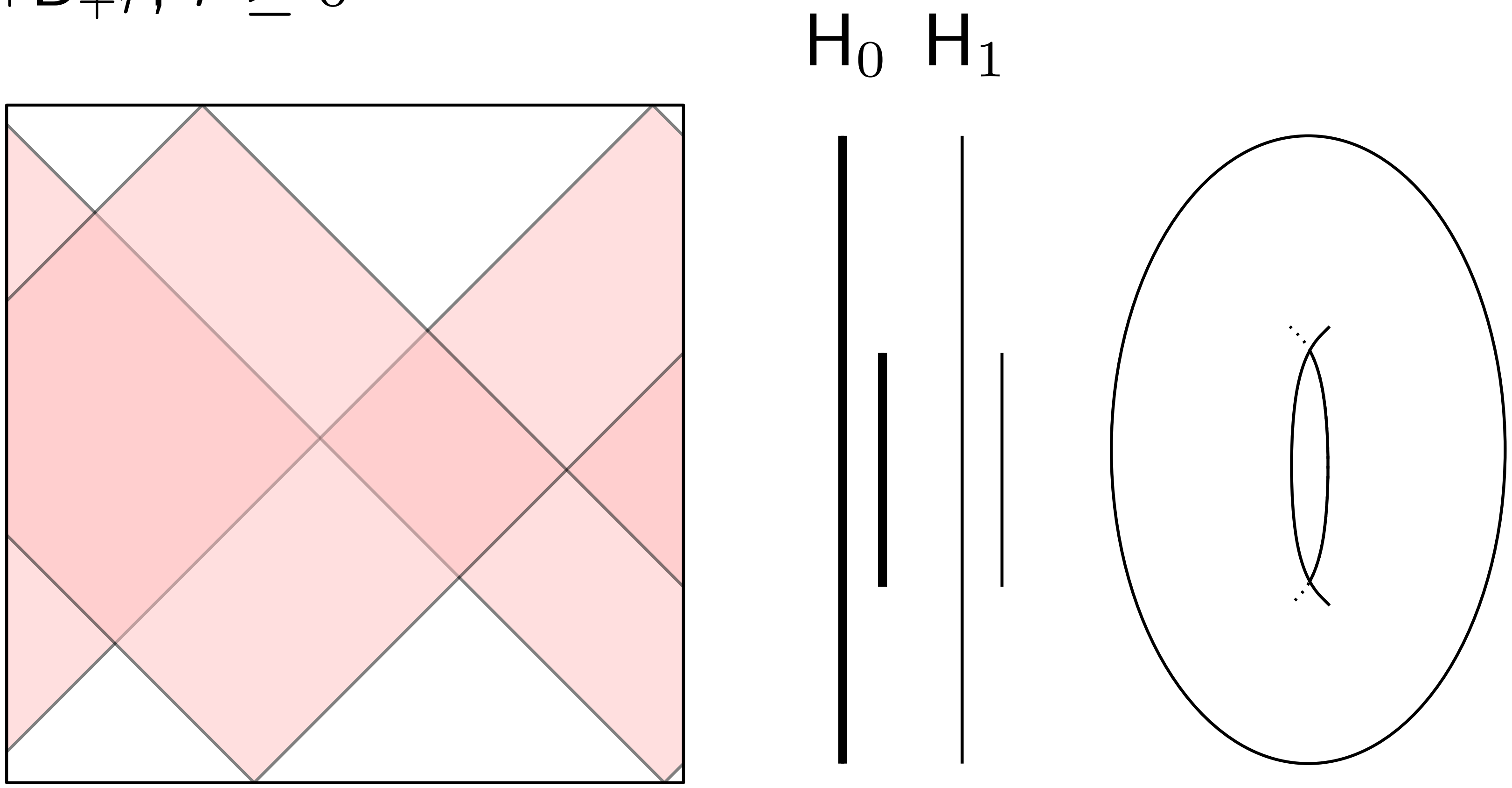
One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a-r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a+r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:

$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$



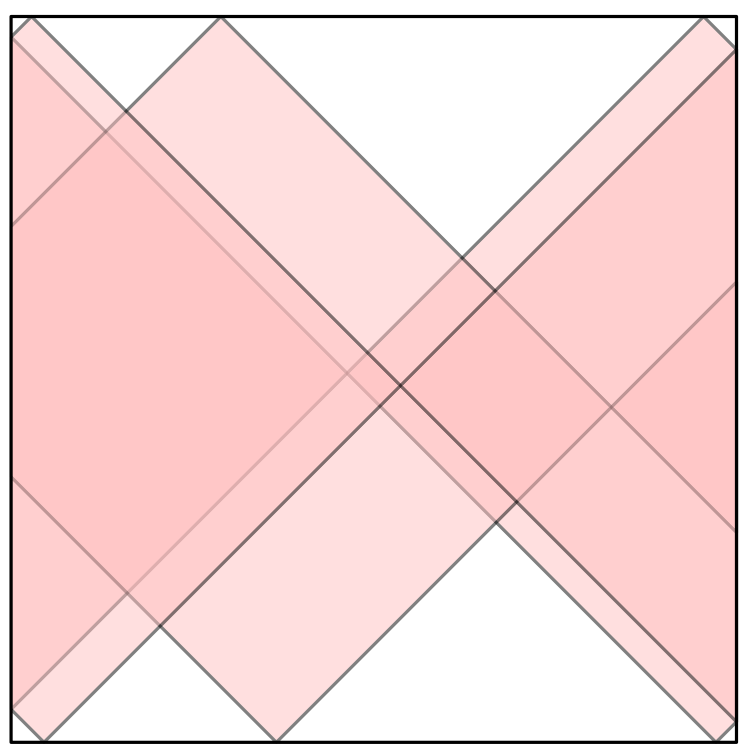
Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

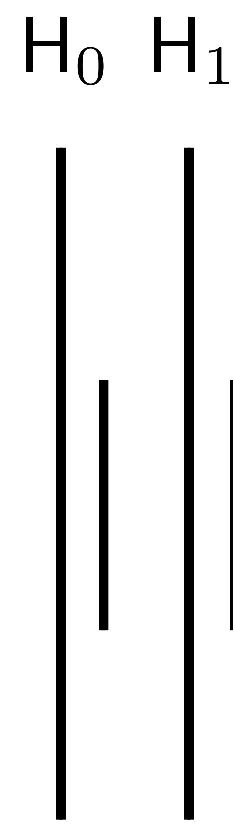
One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

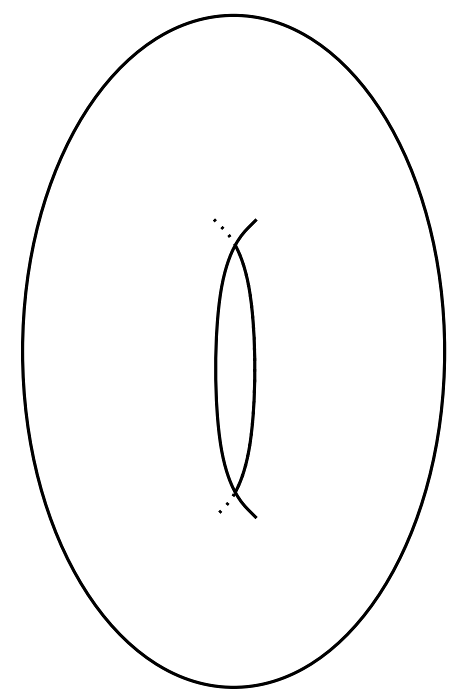
$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a - r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a + r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:
$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$







Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

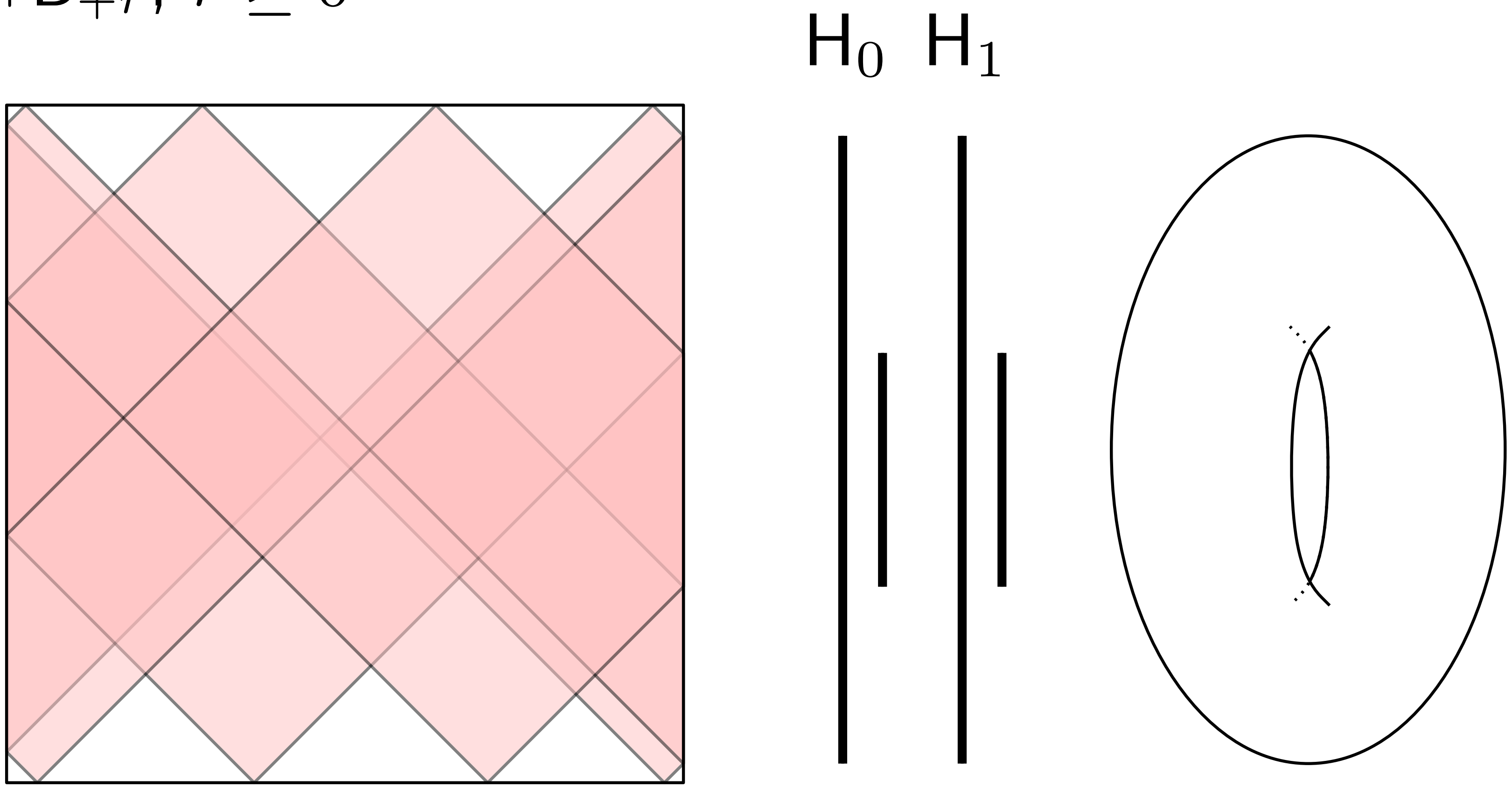
One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a-r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a+r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:

$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$



Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

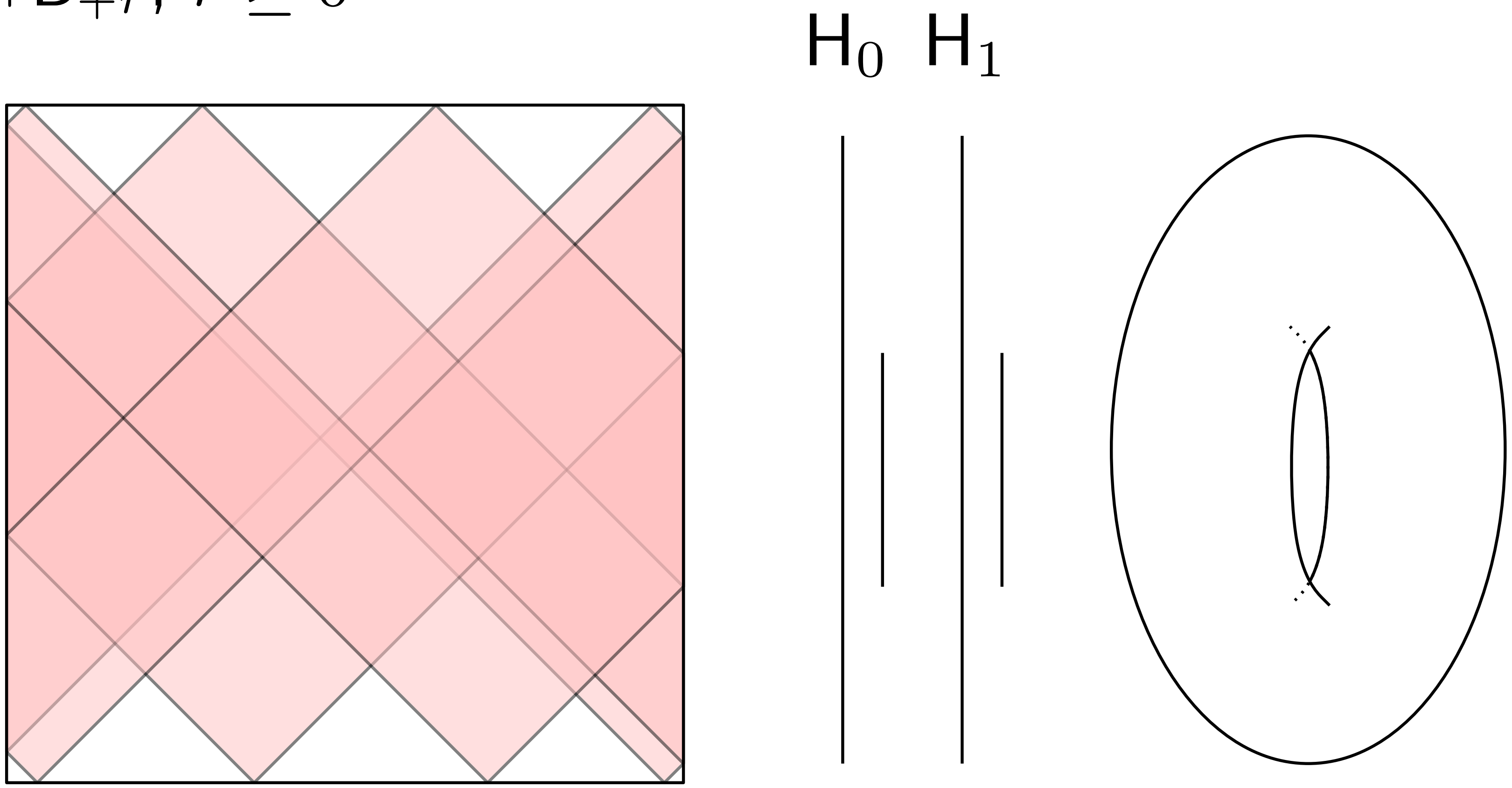
One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a - r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a + r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:

$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$

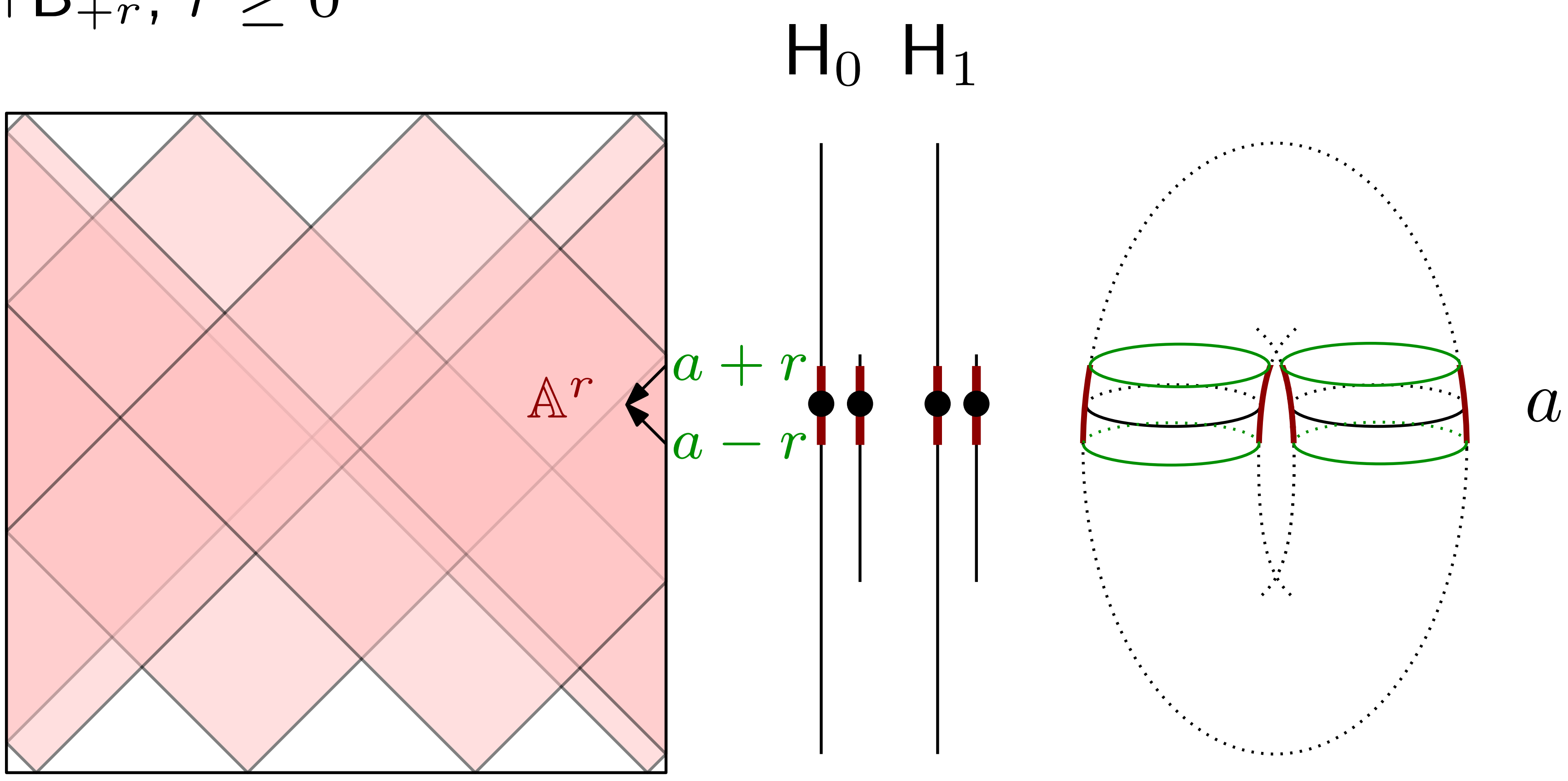


Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a - r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$
$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a + r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

One Point Formula:
$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$

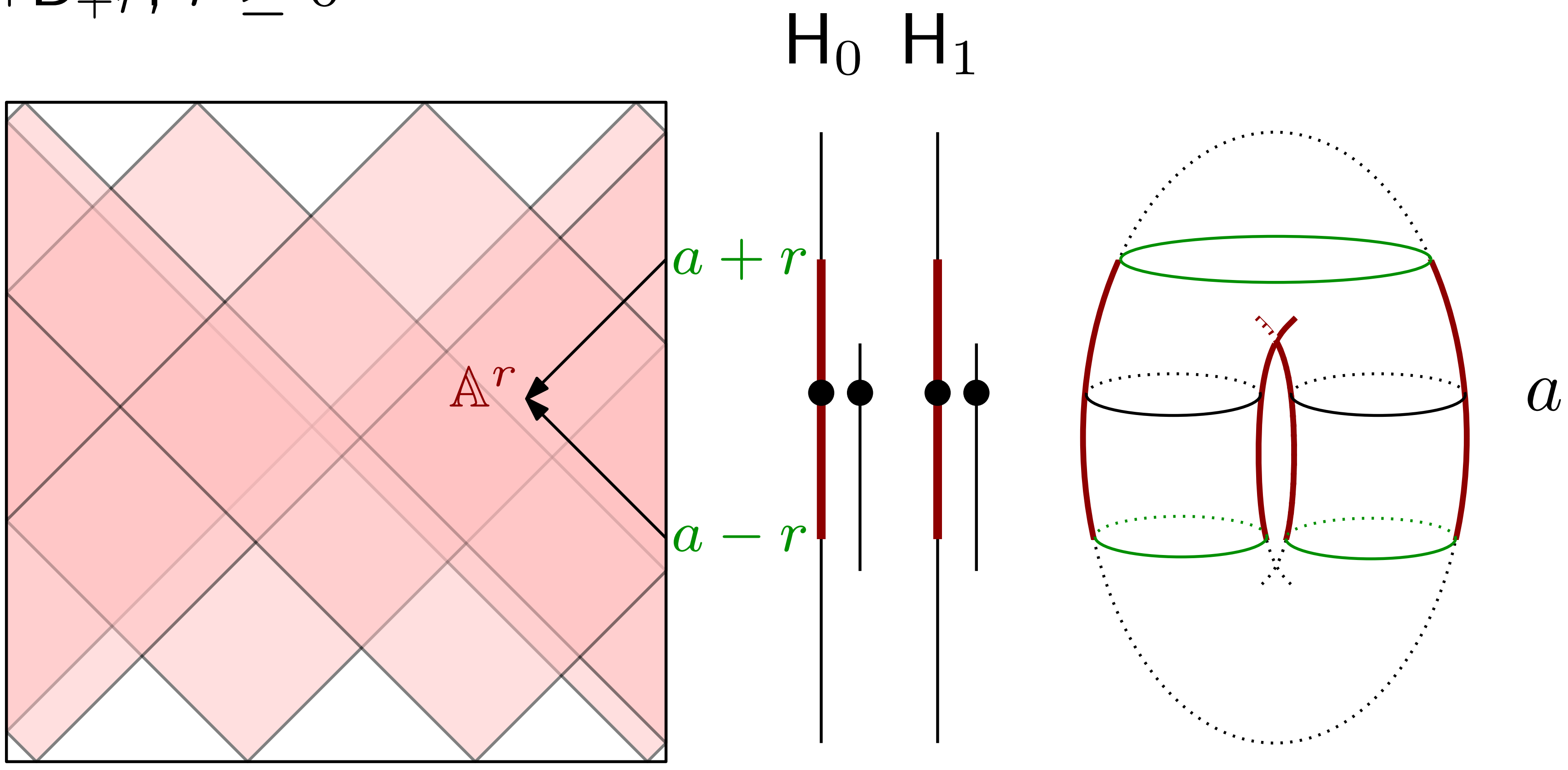


Connection to Persistence ($\mathbb{Y} = \mathbb{R}$)

One point case: $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{A} = \{a\}$ $\mathbb{A}^r = [a - r, a + r]$

$$\mathsf{B}_{-r} = \mathrm{im}(\mathsf{H}(f^{-1}(a - r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$
$$\mathsf{B}_{+r} = \mathrm{im}(\mathsf{H}(f^{-1}(a + r)) \to \mathsf{H}(f^{-1}(\mathbb{A}^r)))$$

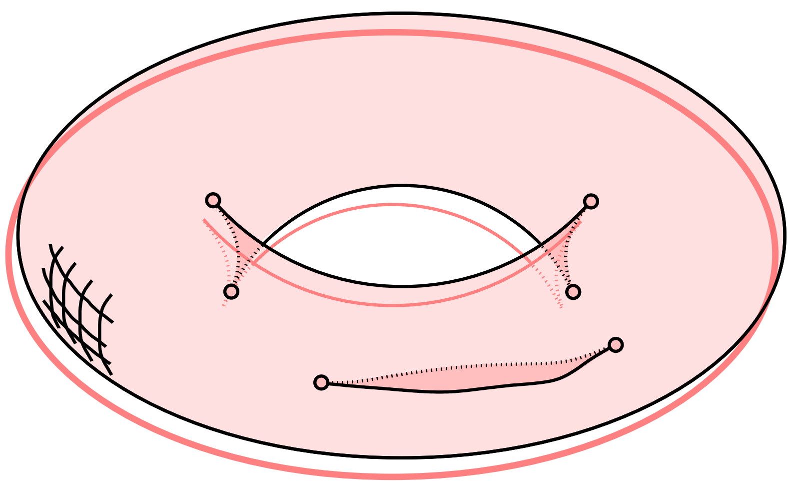
One Point Formula:
$\mathsf{U}(r) = \mathsf{B}_{-r} \cap \mathsf{B}_{+r}$, $r \geq 0$



Well Groups Summary

Stability in a more general setting: $f : \mathbb{X} \to \mathbb{Y}$, $\mathbb{A} \subseteq \mathbb{Y}$

- fixed points
- self-intersections
- contours of a surface



Most algorithmic questions are wide open;
only know how to compute:

$\mathbb{X}$	$\mathbb{Y}$	$\mathbb{A}$
m	m	0
m	1	$*$

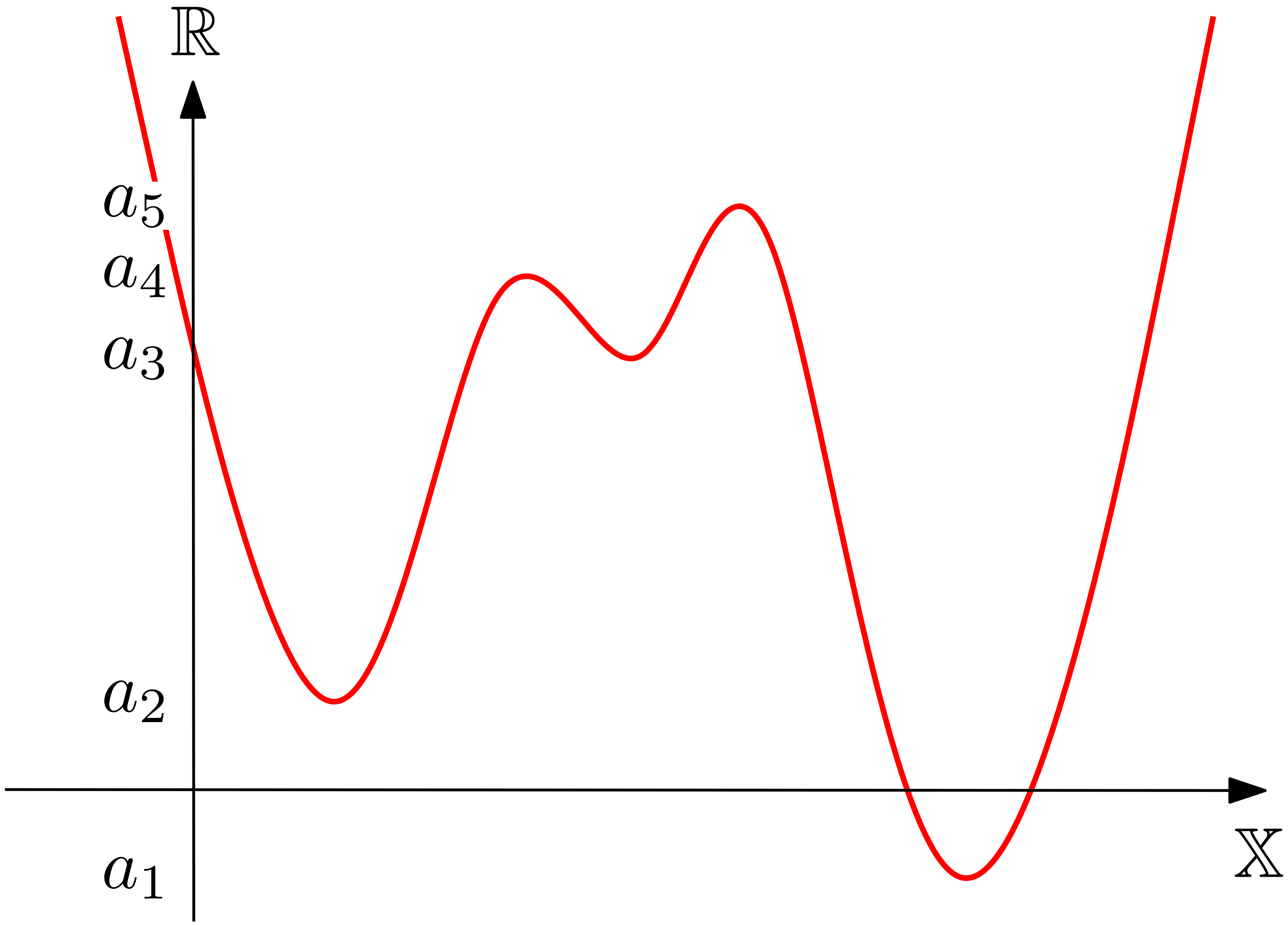
Thank you for your time and attention!

Title

Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

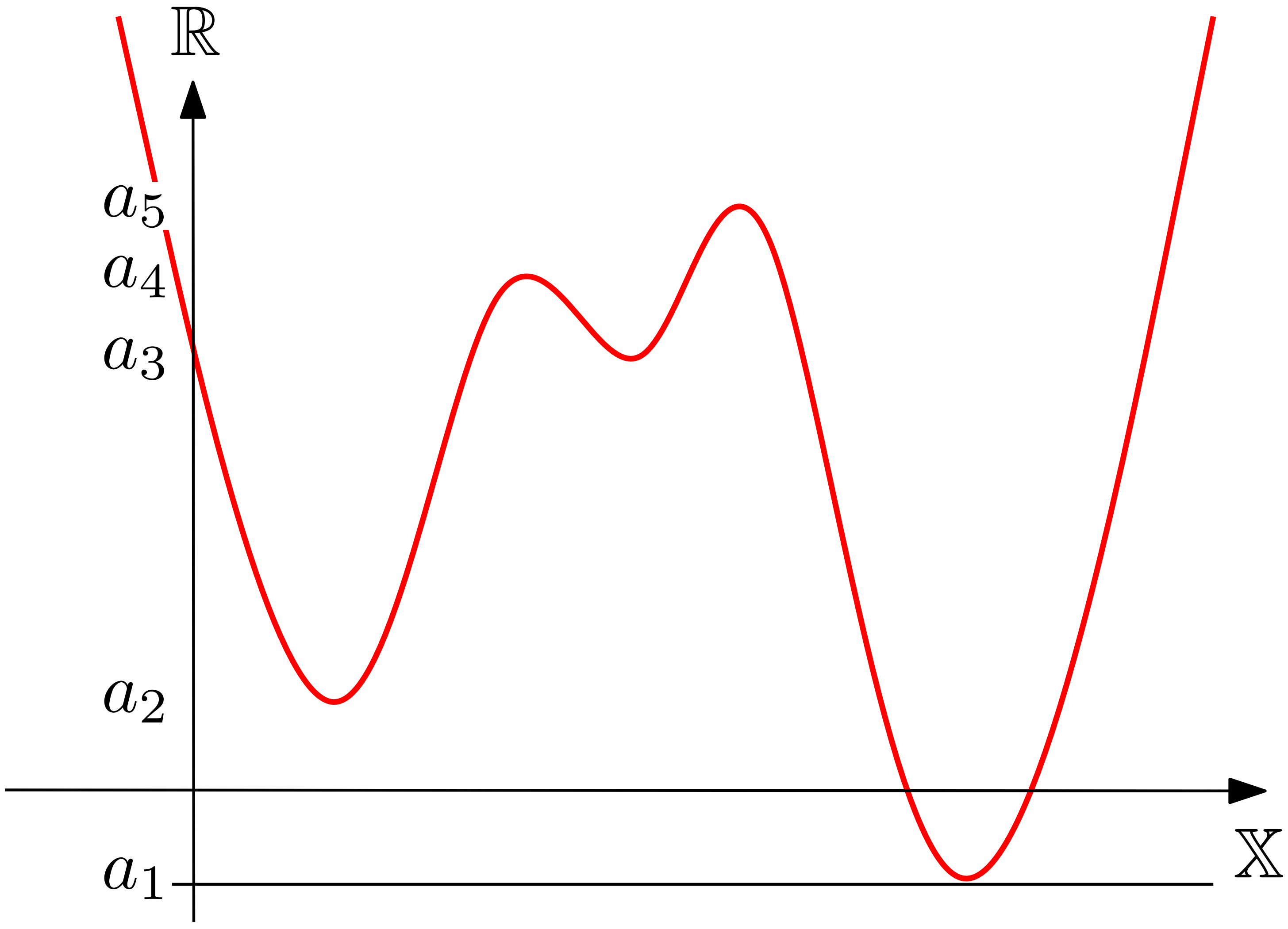
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

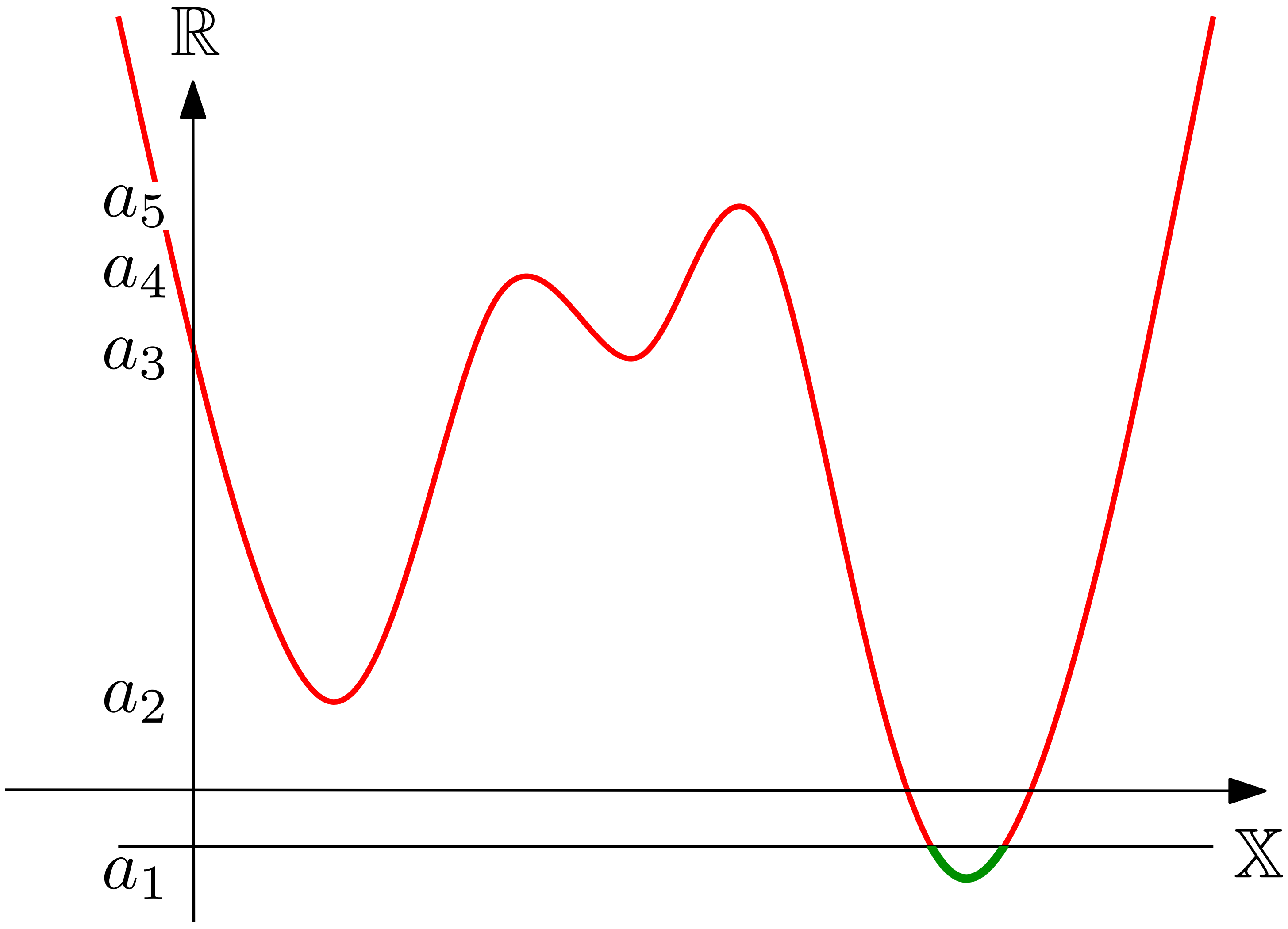
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

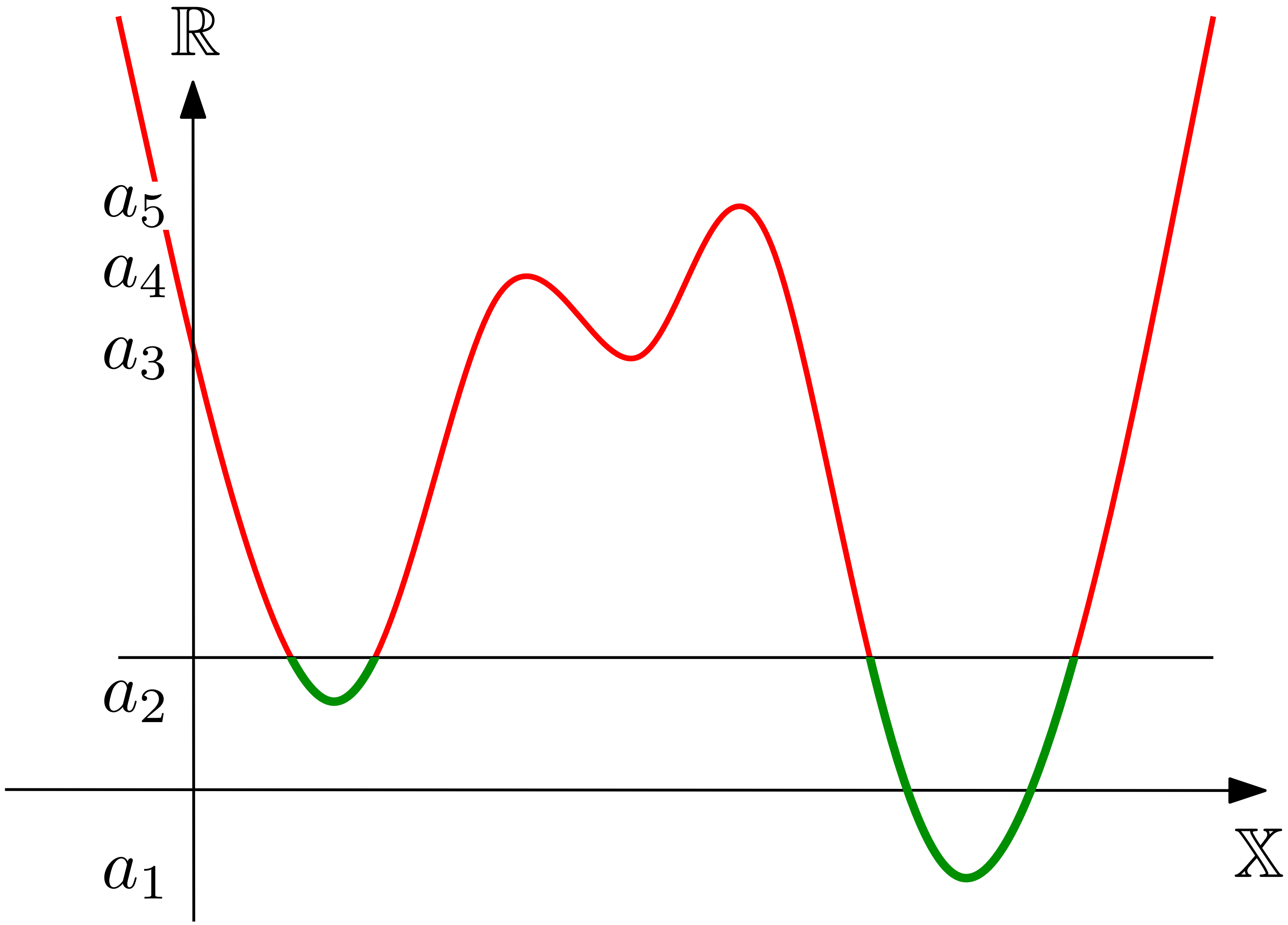
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

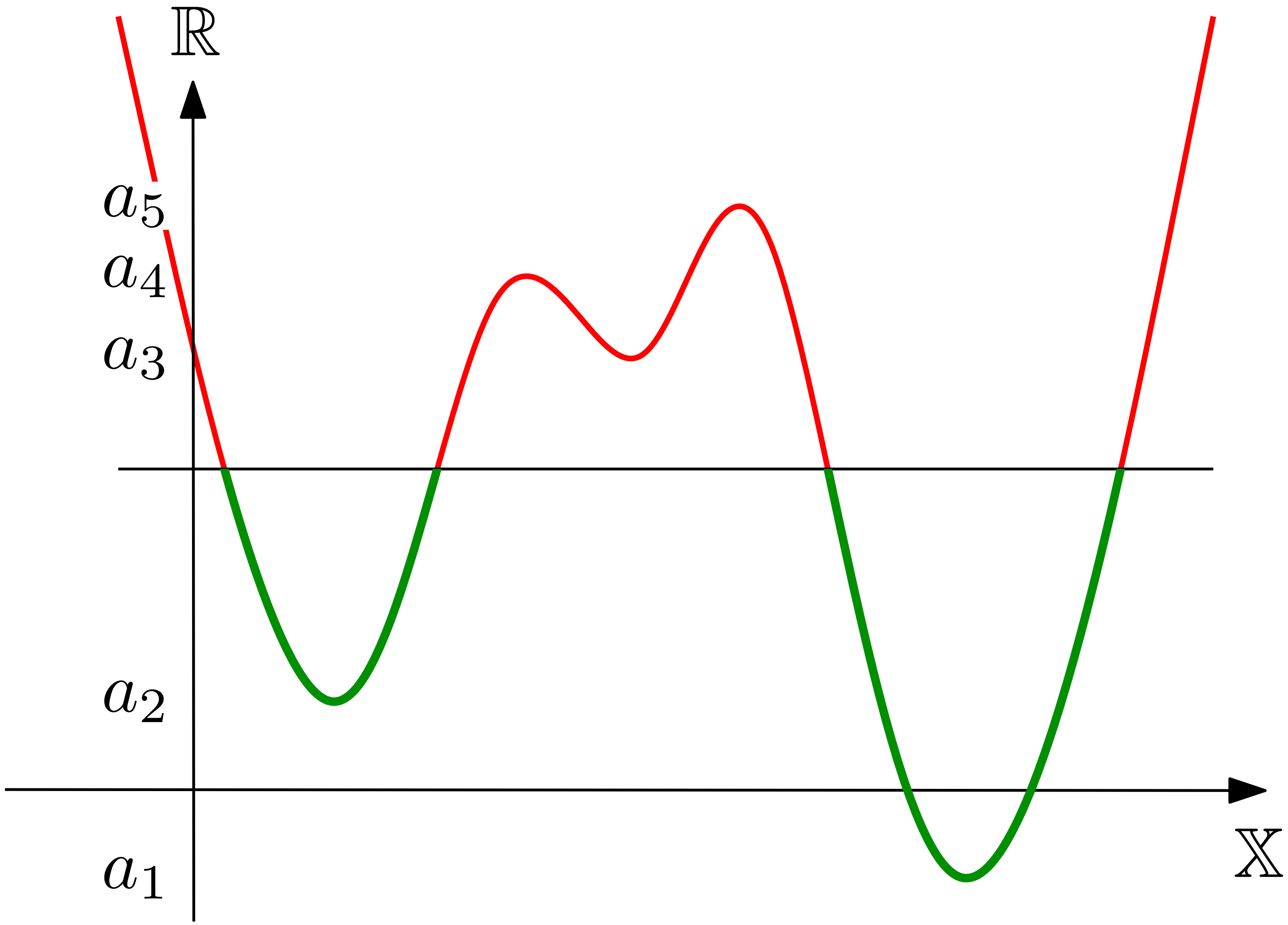
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

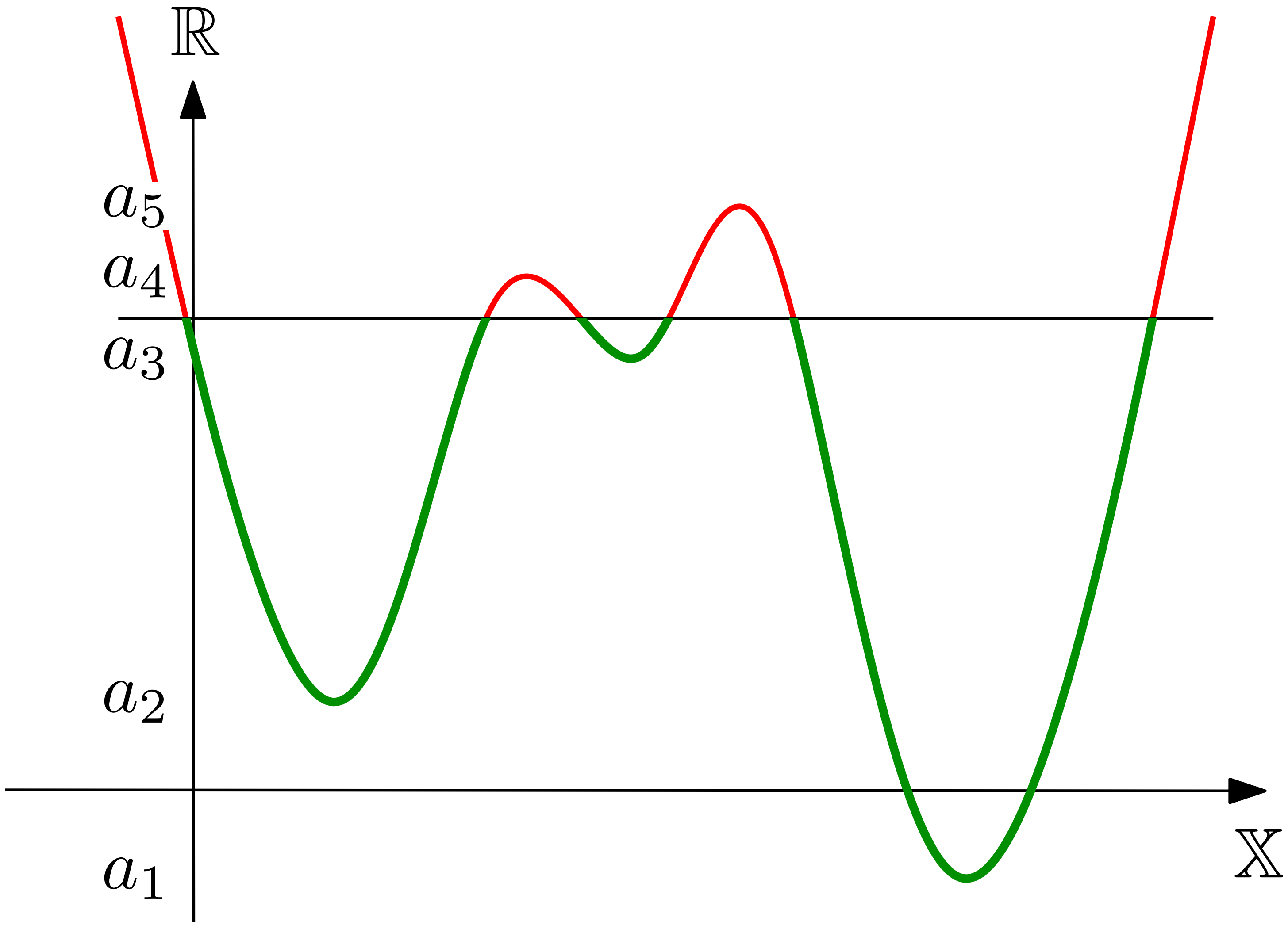
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

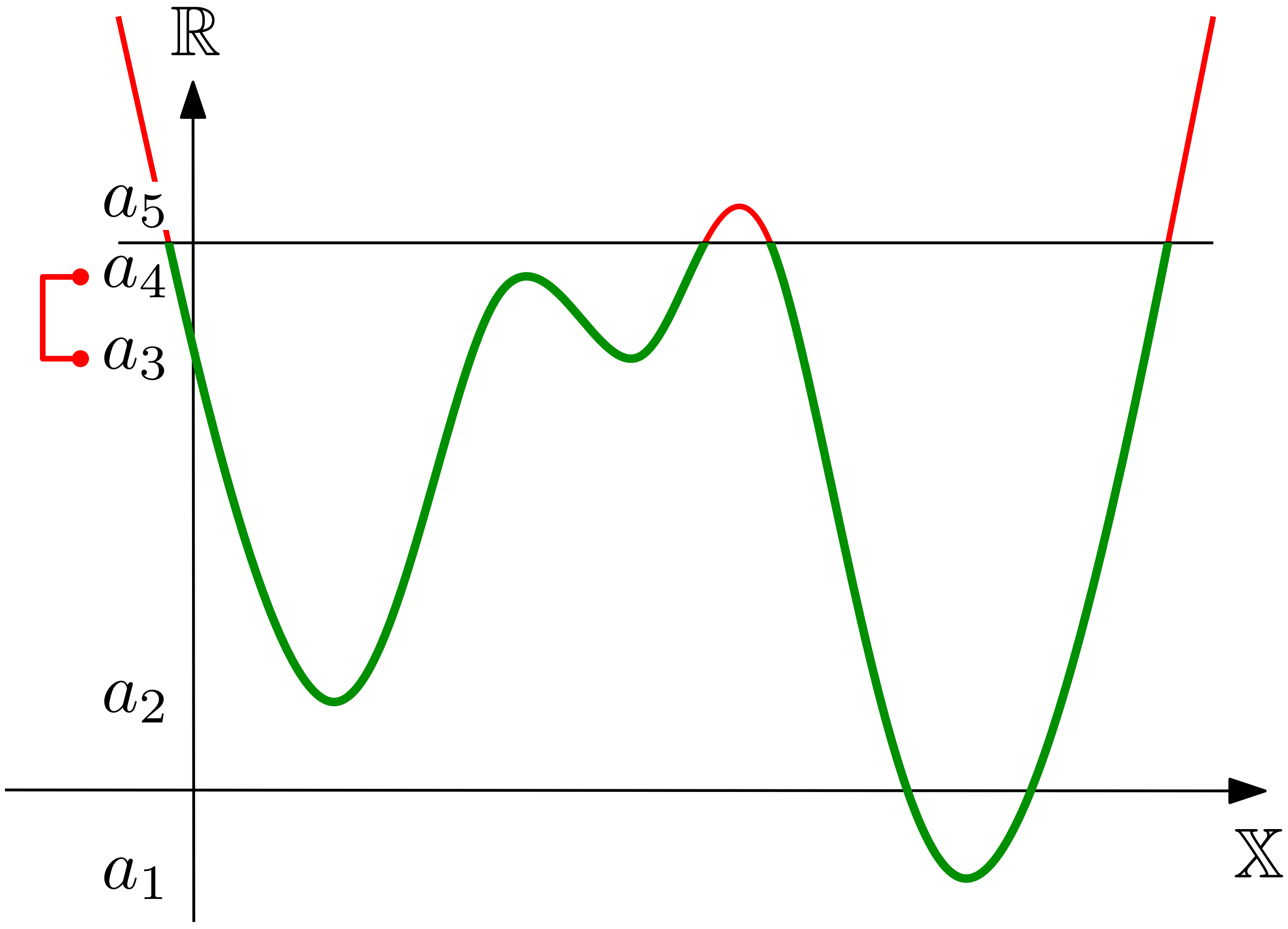
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

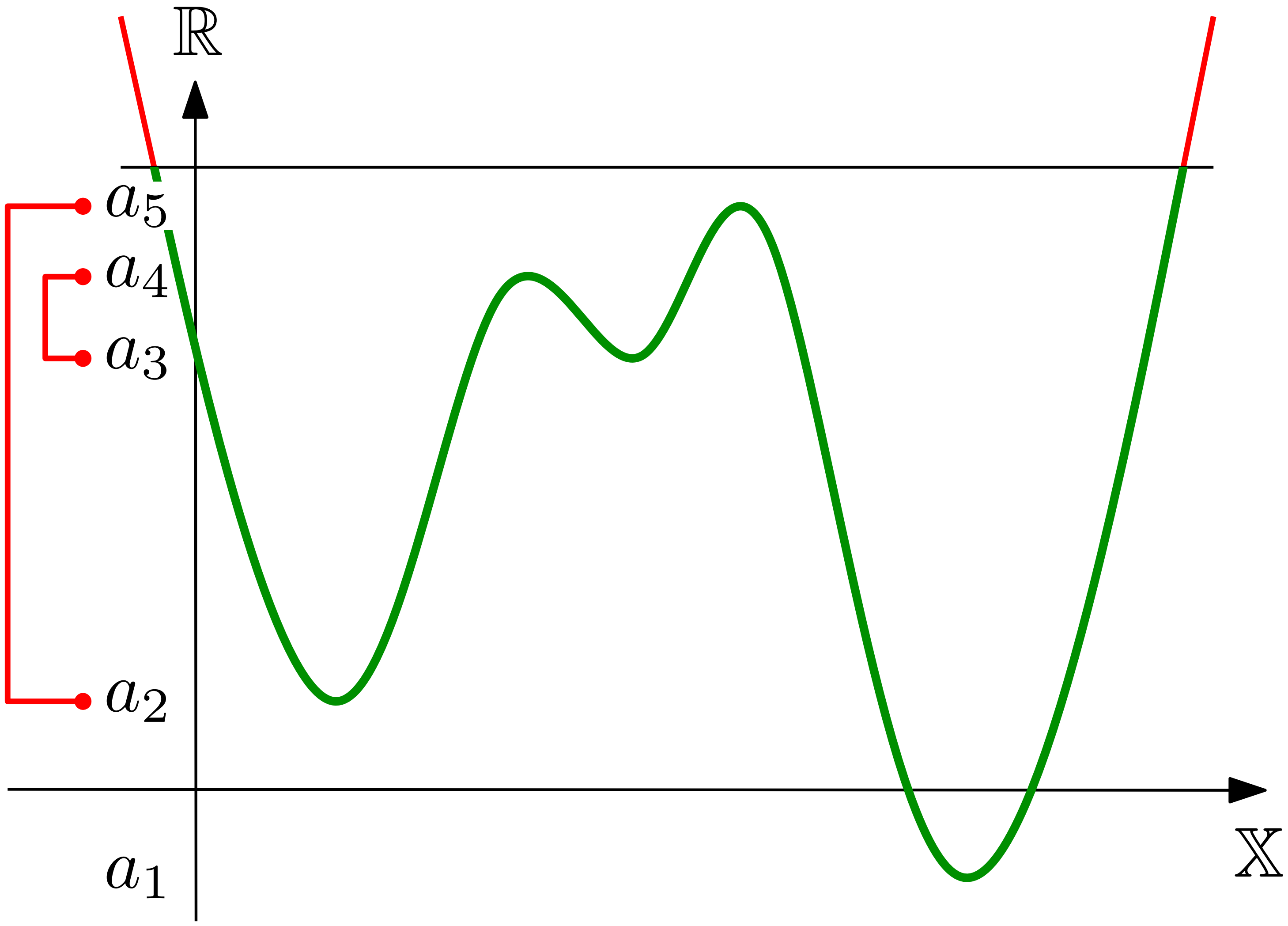
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

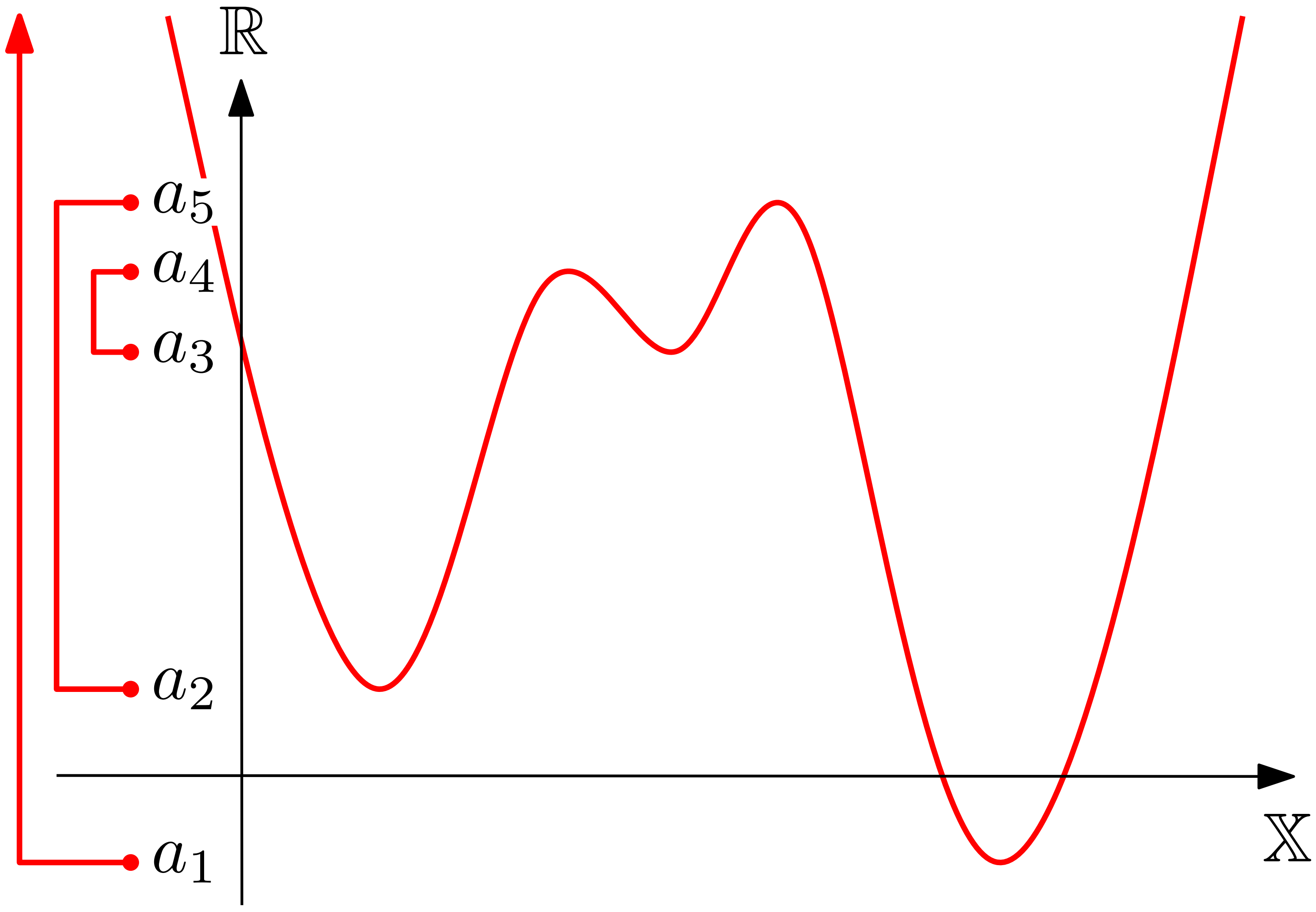
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

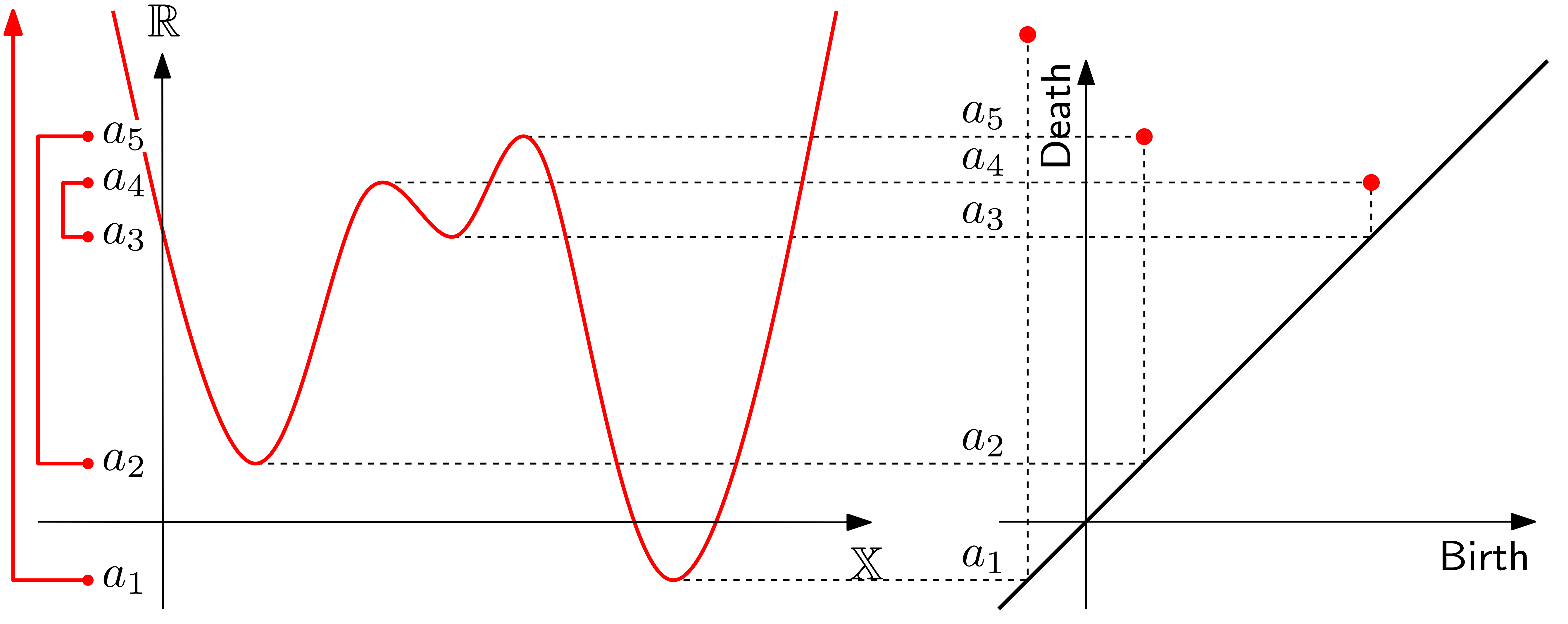
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

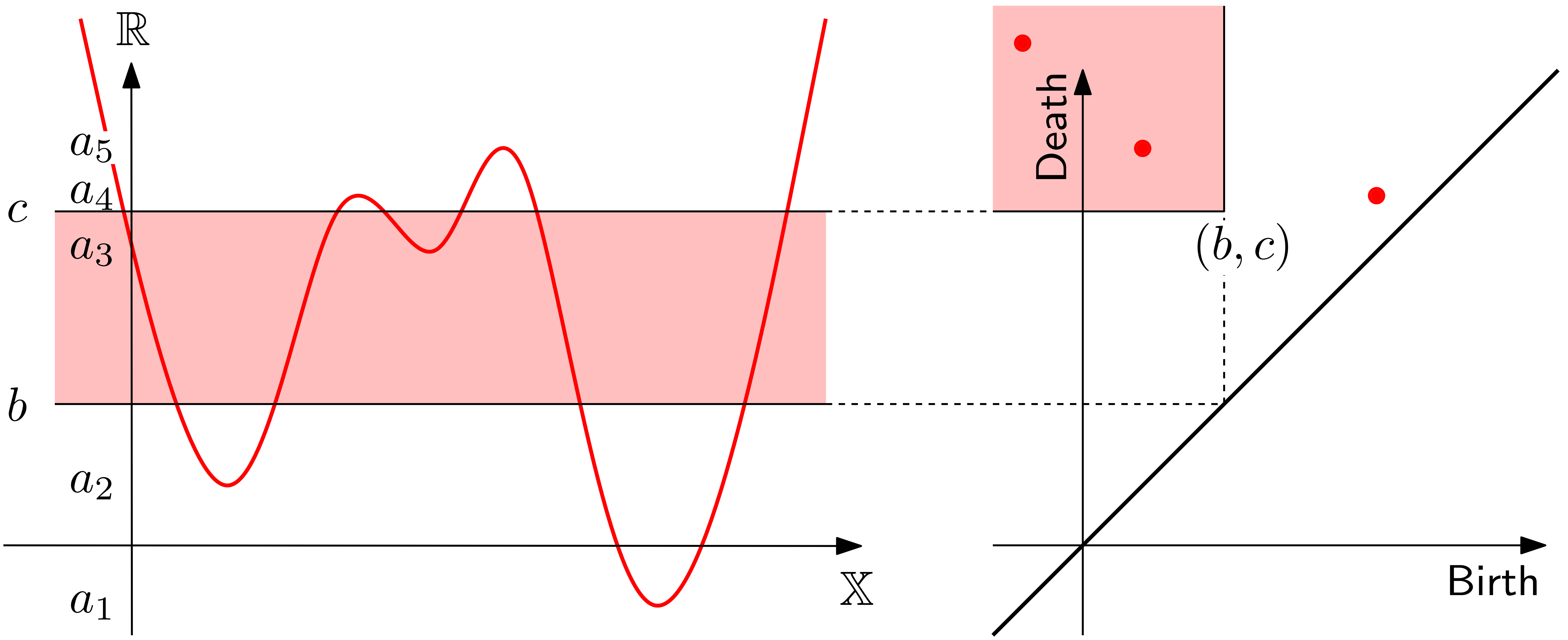
$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



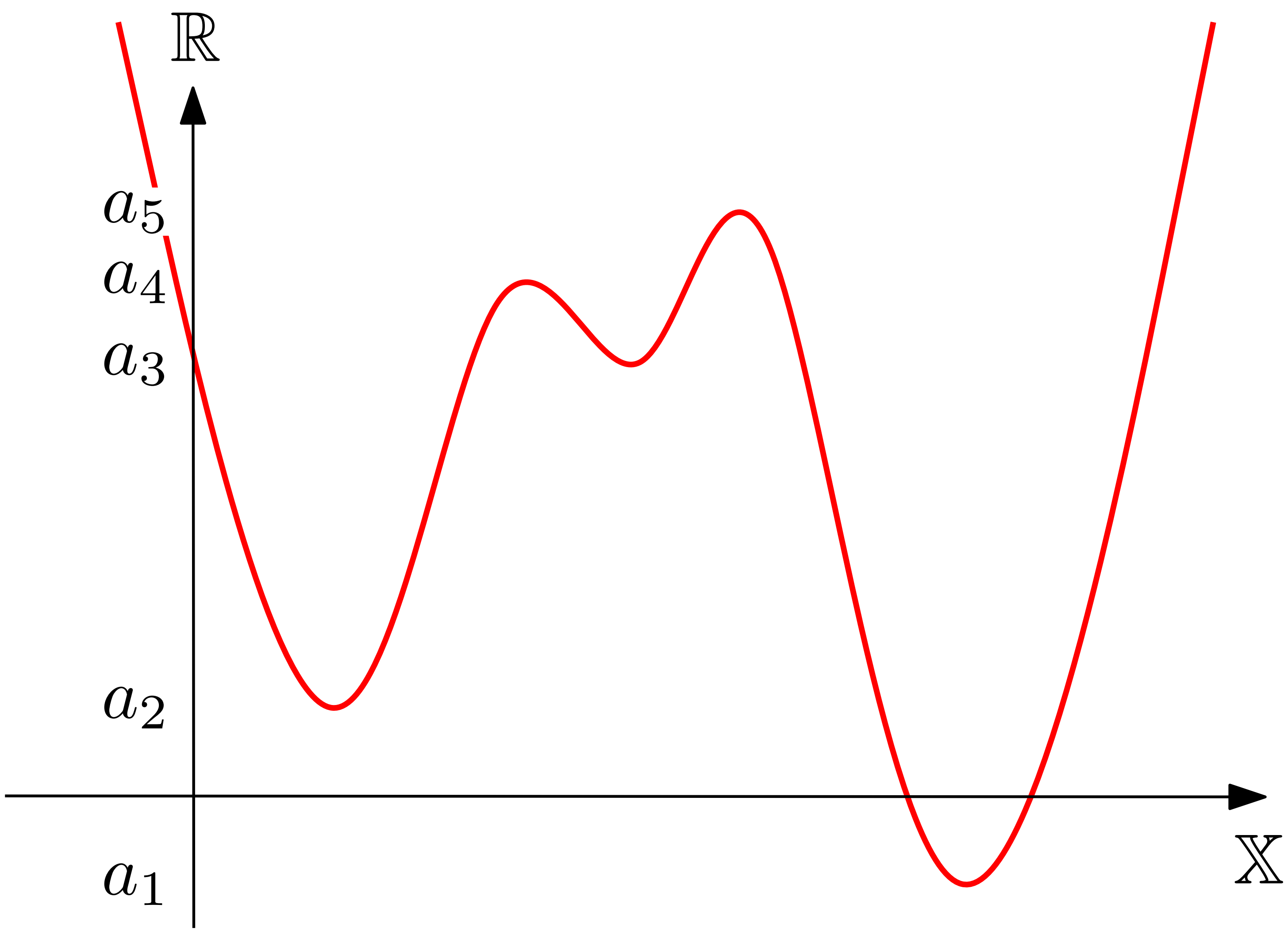
$$\text{rk } \mathsf{H}(\mathbb{X}_b) \to \mathsf{H}(\mathbb{X}_c) = \#^{(b,c)}(\mathsf{Dgm}(f))$$

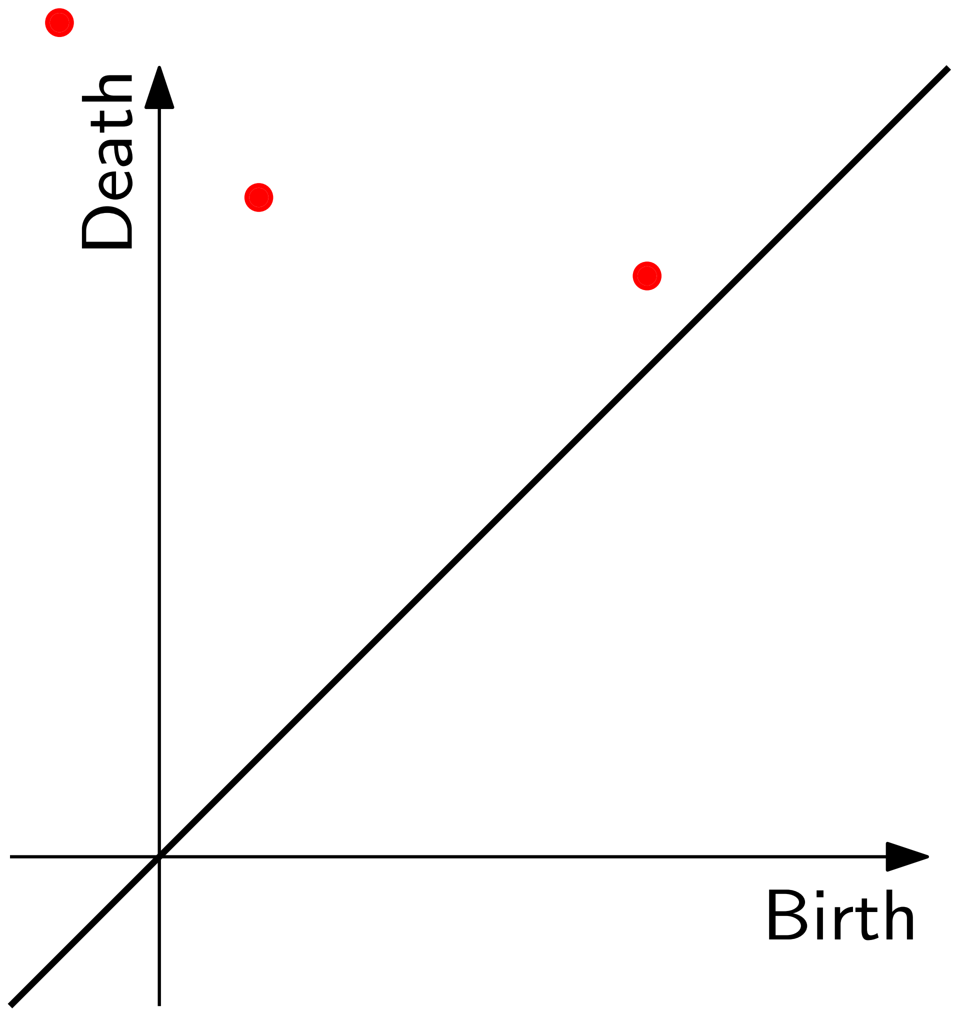
$$= \text{number of points in the upper-left quadrant } (b, c) \text{ of } \mathsf{Dgm}(f)$$

Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$

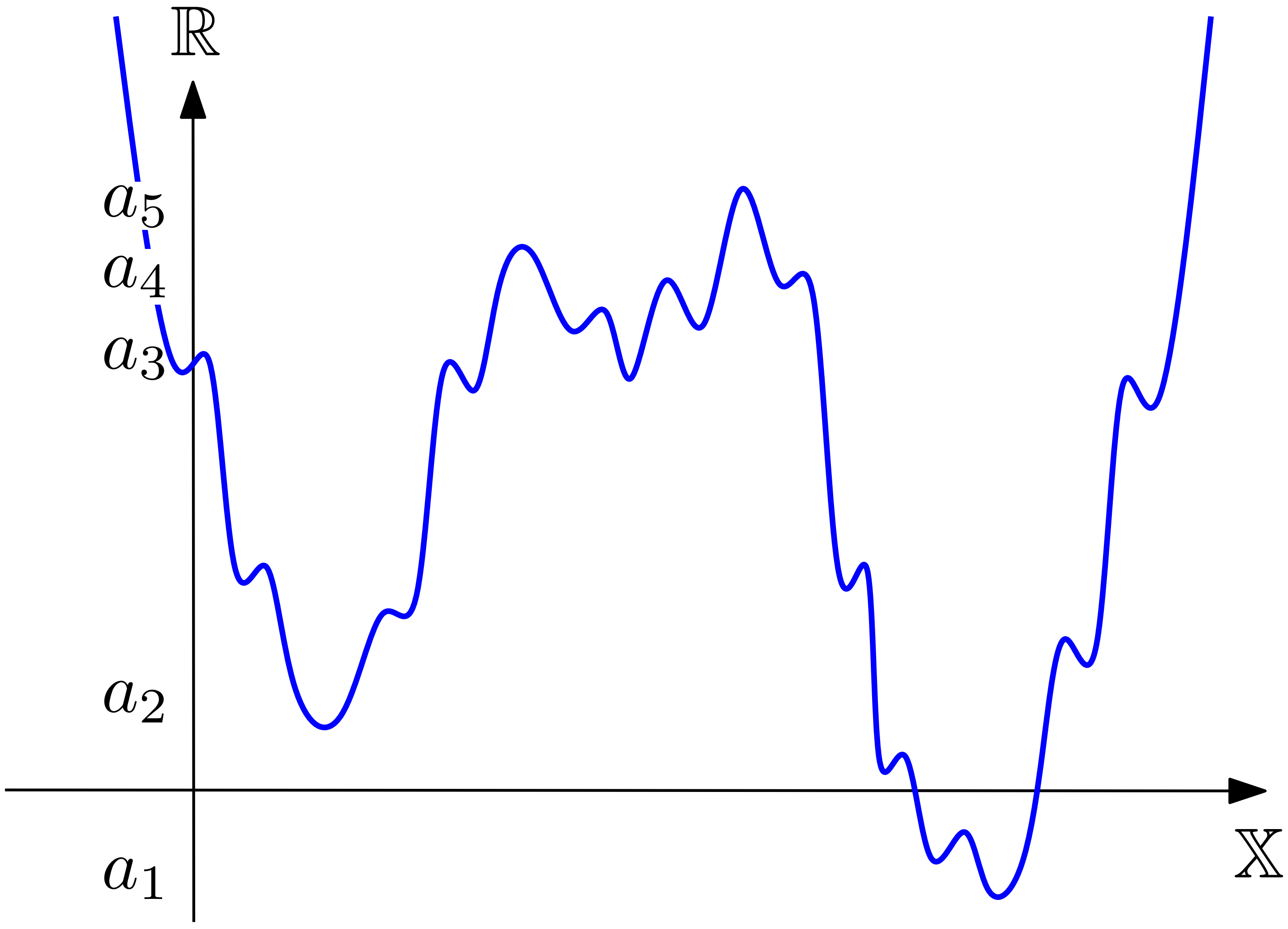


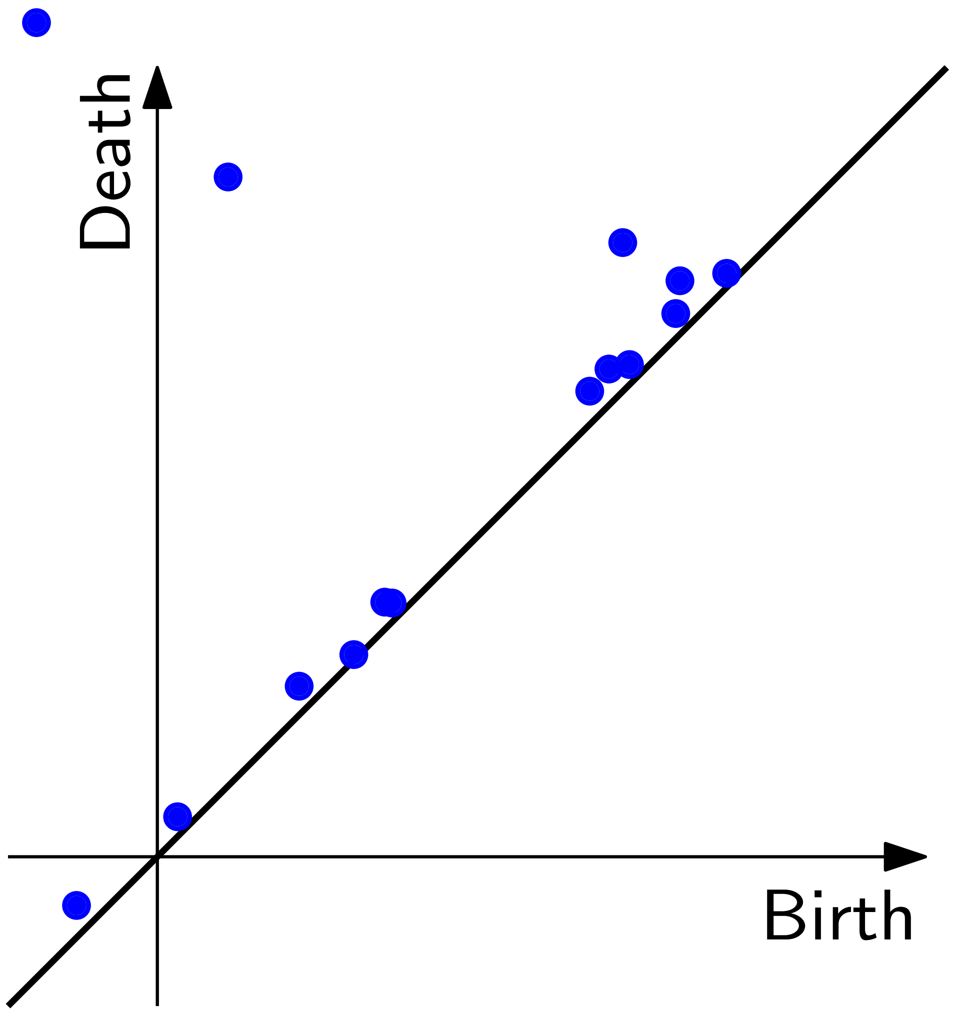


Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$

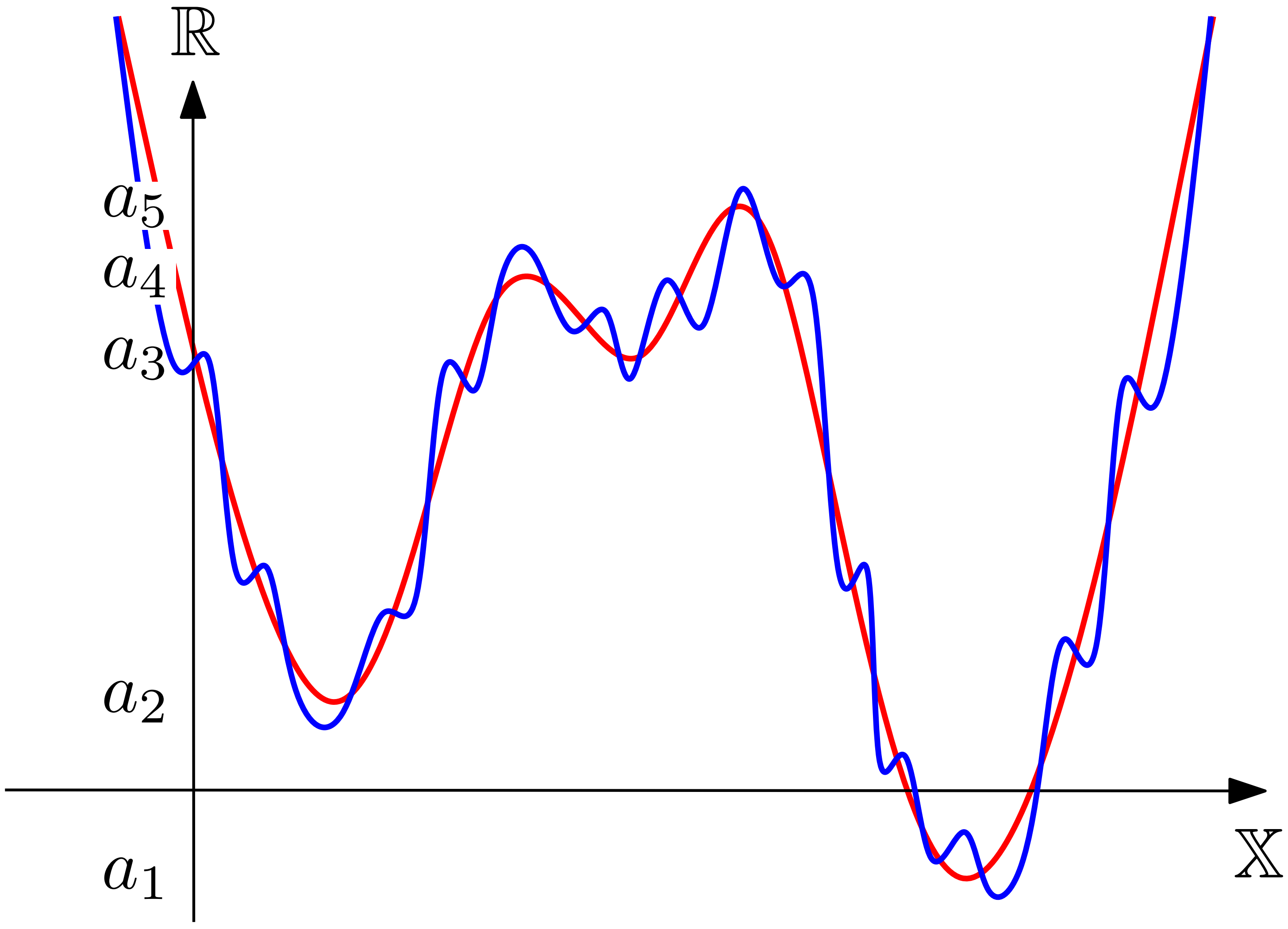


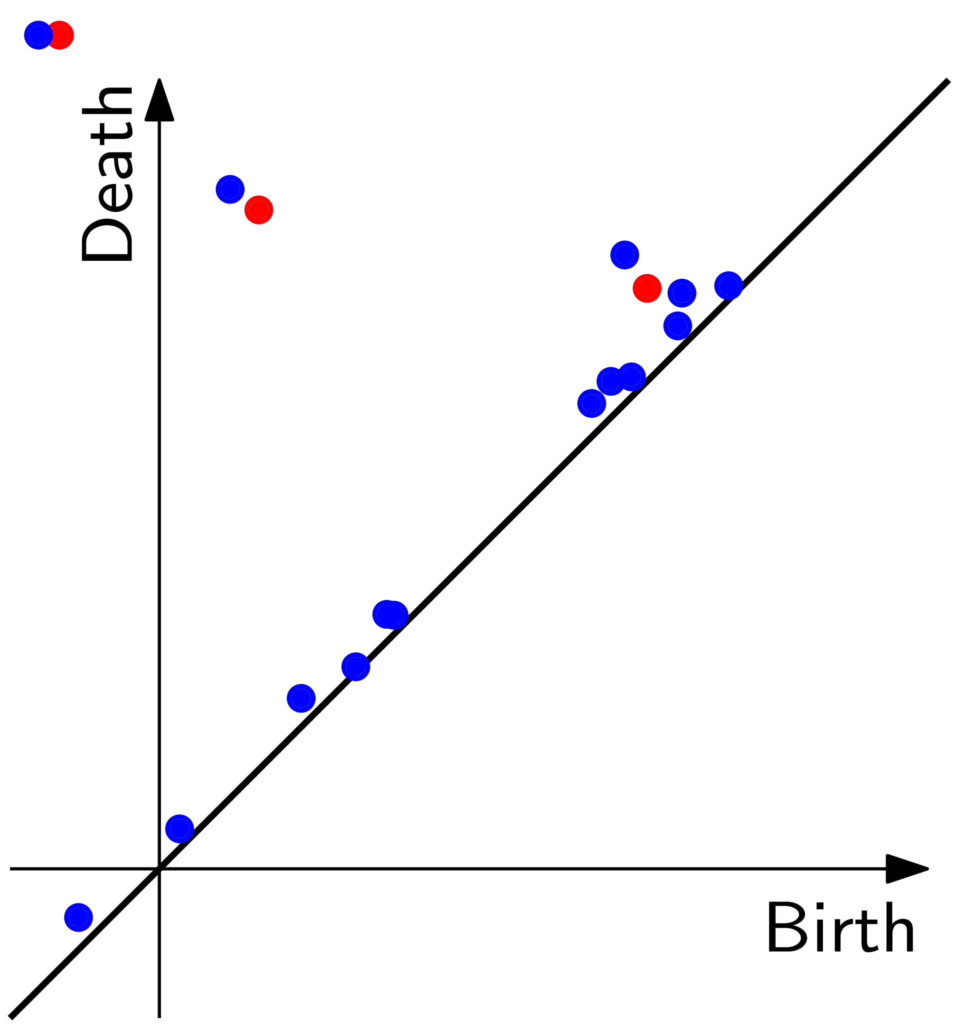


Persistent Homology

$$f : \mathbb{X} \to \mathbb{R}, \qquad \mathbb{X}_a = f^{-1}(-\infty, a]$$

$$0 \to \mathsf{H}(\mathbb{X}_{a_1}) \to \mathsf{H}(\mathbb{X}_{a_2}) \to \ldots \to \mathsf{H}(\mathbb{X})$$



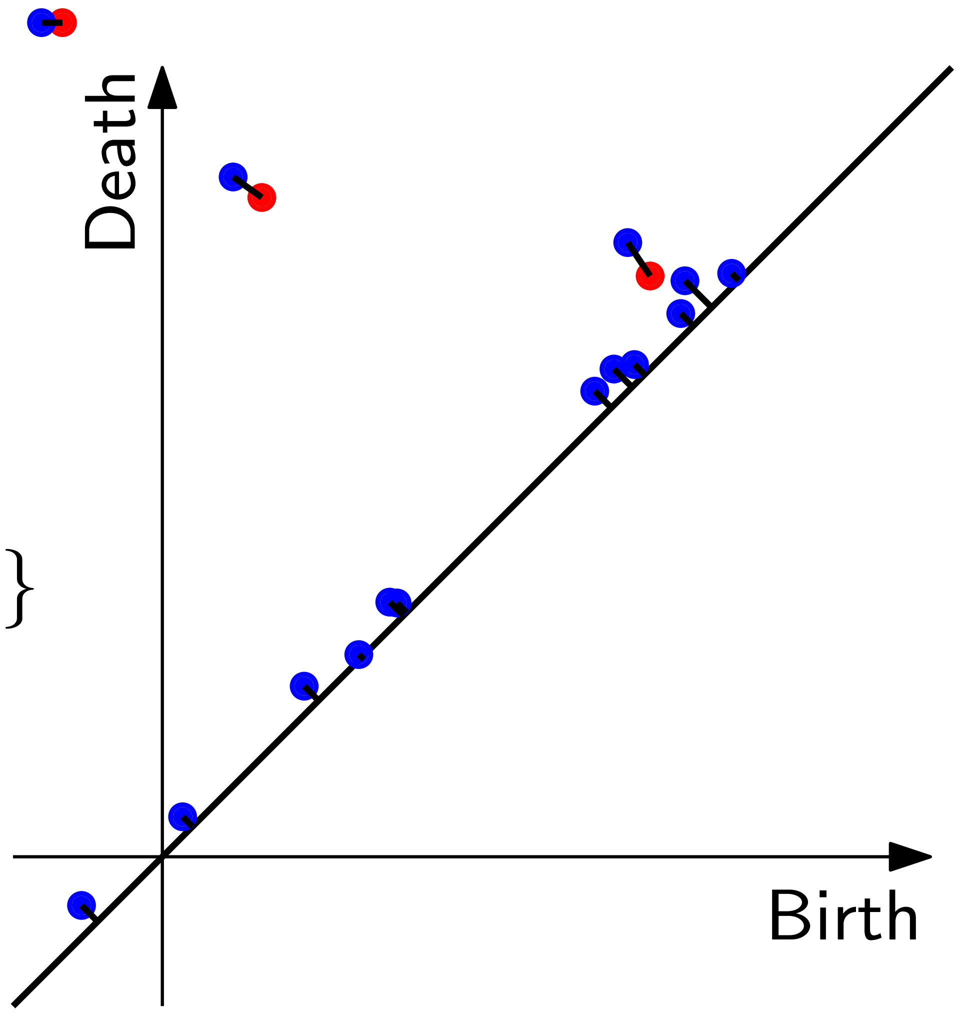


Bottleneck Distance

Bottleneck distance between multisets D and E in $\bar{\mathbb{R}}^2$:

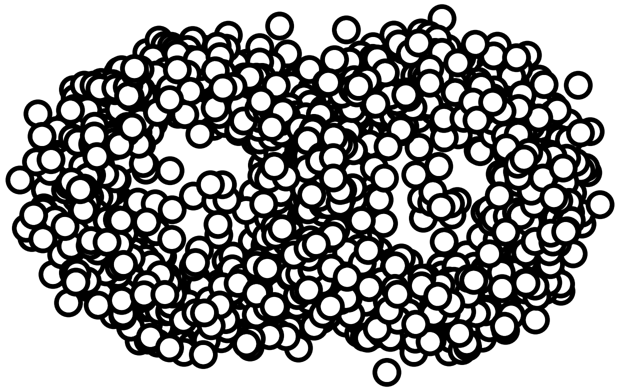
$$W_\infty(D, E) = \inf_\gamma \sup_x ||x - \gamma(x)||_\infty$$

where $x \in D$ and $\gamma \in \{$bijections from D to $E\}$



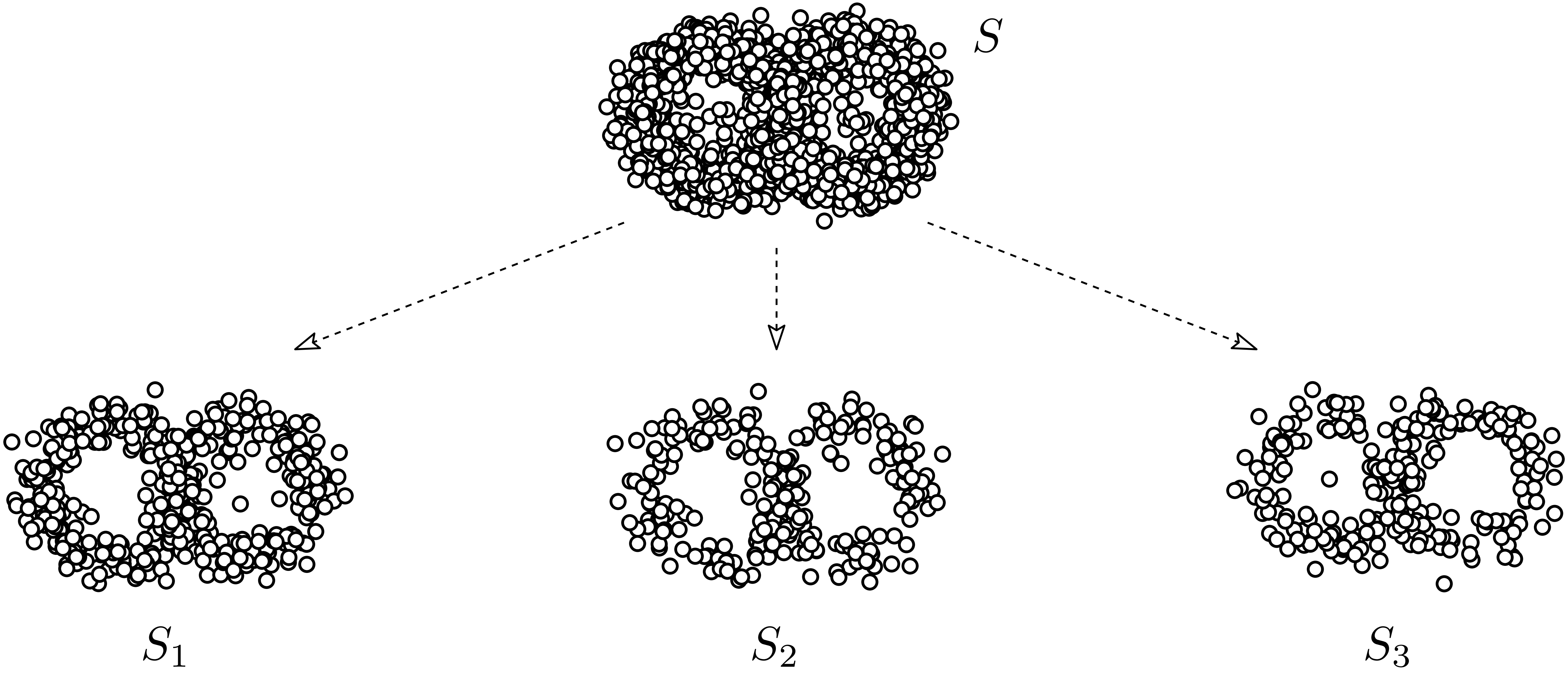
Zigzag Persistence

[Carlsson, de Silva '09]



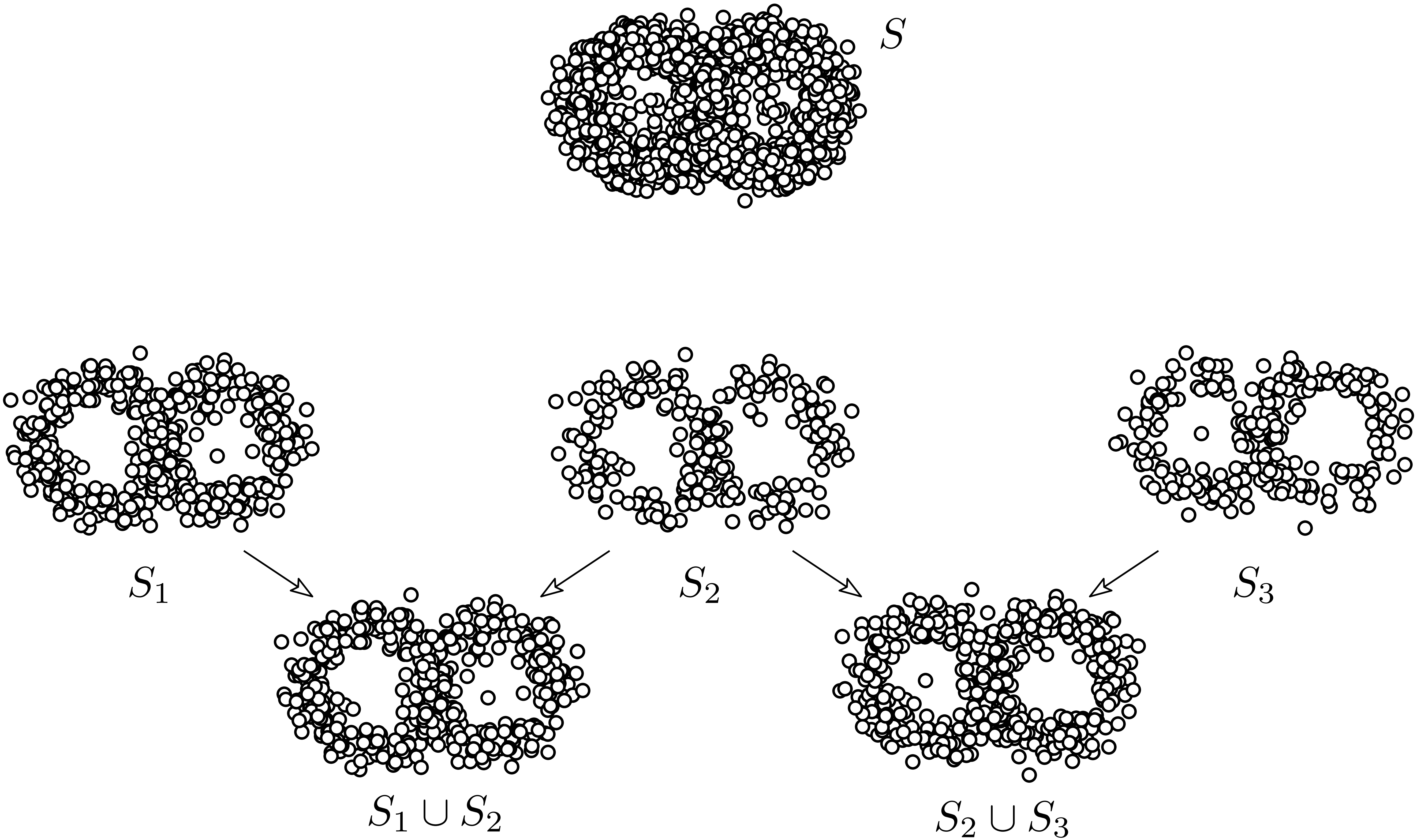
Zigzag Persistence

[Carlsson, de Silva '09]



Zigzag Persistence

[Carlsson, de Silva '09]



Zigzag Persistence

[Carlsson, de Silva '09]