Linking and Caging

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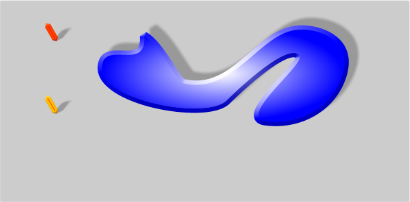
[2]supported by DARPA's SToMP grant.

June 21, 2010



caging

In robotics, *caging* of a (movable) body $\mathcal{D}$ is a configuration of other bodies (obstacles, or, sometimes, *fingers*) which restrict the motions of $\mathcal{D}$ to a bounded region. For example, *immobilizing* is a special case of caging.

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We will restrict here to planar objects: the body $\mathcal{D}$ is a planar bounded compact set which we will assume to have piece-wise smooth boundary (or semi-algebraic or polygonal...).

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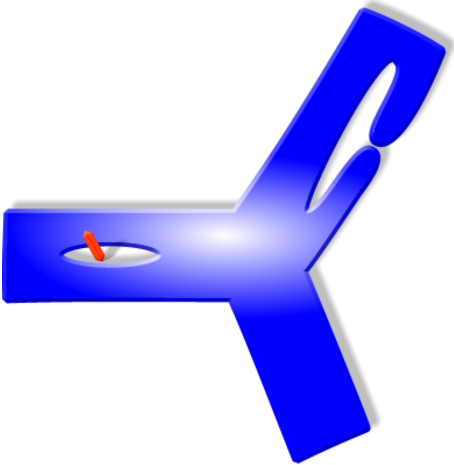
The obstacles in this talk are mostly points; more general cases can be handled without much overhead. (In the simplest situation when the bounding bodies are round disks of the same radius, one can replace $\mathcal{D}$ by its parallel body.)

caging

Example of caging with one finger



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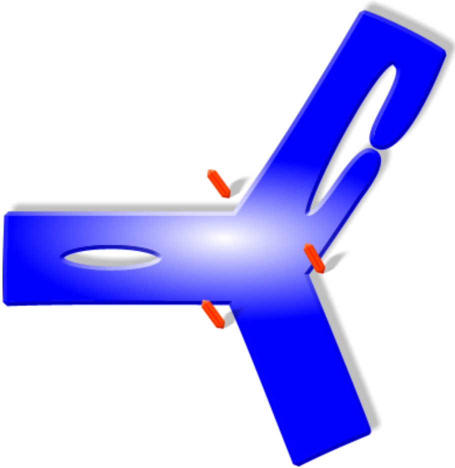
Example of caging with three fingers



caging as a topological problem

This talk deals with the topological aspects of caging.

Most important type of questions asked about such spaces question is about the structure of their connected components: there is a distinguished component corresponding to all obstacles far away from the body, and all other components correspond to caging configurations.

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Topological aspects of caging were addressed, implicitly, in the beautiful '89 paper by Goodman, Pach and Yep on *Mountain Climbing, Ladder Moving, and the Ring-Width of a Polygon*; more recently by Rimon and Blake ('96) and Mason and Rodriguez ('08).

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We start with some warm-up stories.

on cars and wagons

Consider the following (well-known) problem from *V.I. Arnold's* book on ordinary differential equations:

Problem 1. Two nonintersecting roads lead from City A to City B (Fig. 1). Suppose it is known that two cars connected by a rope of length less than $2l$ manage to go from A to B along different roads without breaking the rope. Can two circular wagons of radius l whose centers move along the roads in opposite directions pass each other without colliding?

Solution. Consider the square

$$M = \{(x_1, x_2) : 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$$

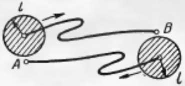
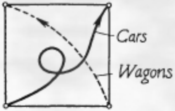


Fig. 1 Initial position of the wagons.



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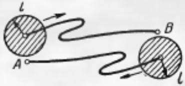

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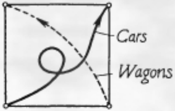

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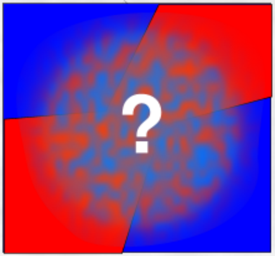


on cars and wagons, cont'd

We see a certain dichotomy: either the cars, or the wagons can do the job.

The problem is essentially topological (despite its geometric appearance): the obstacle to two red corners belonging to the same connected component is the fact that two blue corners are in the same component, and vice versa.

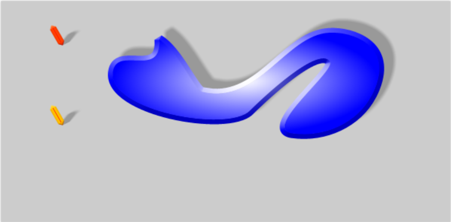
> *There is an obvious resemblance to the dualities of the optimization problems:* max *of the primary functional is equal to* min *of the functional of the dual problem.*

squeezing a camel through the eye of a needle

Let us consider the following problem related to caging.

We start with the planar domain $\mathcal{D}$ (with piece-wise smooth or semialgebraic boundary) homeomorphic to a open disk, and two point configuration $\mathbf{C} = \{p_1, p_2\}$ on plane.

Problem 1 *Can one* squeeze $\mathcal{D}$ *between the point configuration* $\mathbf{C}$*?*



pulling the needle around the camel

Equivalently, can one pull the points p_1, p_2 (preserving the distance between them) around $\mathcal{D}$?

Let $\mathcal{E}$ be the group of Euclidean motions of the plane.

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Definition 1 Pulling **C** *around* $\mathcal{D}$ *is a loop* π *in* $\mathcal{E}$ *(that is a continuous mapping* $S^1 \to \mathcal{E}$*) such that the loops* πp_i *in* $\mathbb{R}^2$ *do not meet* $\mathcal{D}$ *and the following "winding" condition holds: index of the loop* πp_1 *with respect to* $\mathcal{D}$ *is* 1*, and the index of* πp_2 *is* 0*.*

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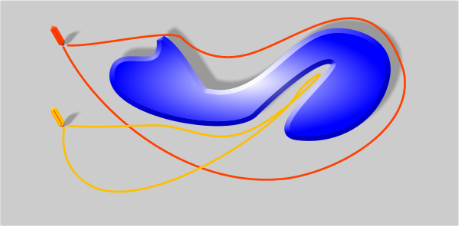

dichotomy

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Proposition 1 *Either one can pull two point configuration* $\mathbf{C}$ *around* $\mathcal{D}$*, or there exists a full rotation of* $\mathbf{C}$ entirely within $\mathcal{D}$*, that is a loop* $\pi' : S^1 \to \mathcal{E}$ *such that the vector* $\pi'_\theta p_1 - \pi'_\theta p_2$ *turns around the origin (perhaps, several times), and both loops* $\pi'_\theta p_1, \pi'_\theta p_2$ *stay within* $\mathcal{D}$*.*

Euclidean motions of the plane

The topology of $\mathcal{E}$ is important: $\mathcal{E}$ is diffeomorphic to the (open) solid torus,

$$\mathcal{E} \cong \mathbb{R}^2 \times S^1 :$$

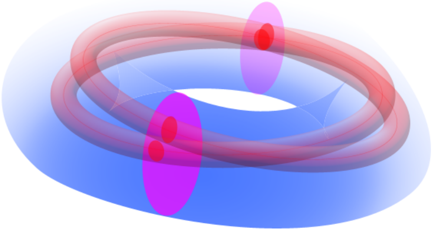
indeed, an orientation-preserving Euclidean motion of the plane can be uniquely represented as a composition of a parallel translation (i.e. $\mathbb{R}^2$), and a rotation (i.e. S^1).

Remark 1 *In fact, one can work with somewhat more convenient 3-dimensional sphere, $S^3 \supset \mathcal{E}$. Indeed, one could without restricting generality first pass to the closed solid torus (by restricting admissible motions to those which do not take p_1 outside of some pre-specified disk $B(R)$ of large enough radius R), and then by allowing the configurations with $p_1' \in \partial B(R)$ and $|p_2' - p_1'| \leq |p_1 - p_2|$. The resulting configuration space is homeomorphic to S^3 and the notion of "pulling through" remains unaffected.*

useful subsets of $\mathcal{E}$

Definition 2 *We will use the following subsets:*

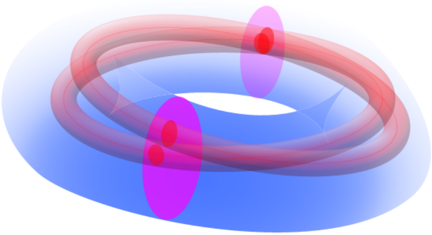
- $O_i \subset \mathcal{E}, i = 1, 2$ *is the set of motions taking* p_i *to* $\mathcal{D}$;
- $F_i := \mathcal{E} - O_i$;
- $O_{12} := O_1 \cap O_2$; $F_{12} := \mathcal{E} - O_{12}$;
- $O := O_1 \cup O_2$; $F := \mathcal{E} - O$.



some properties or F and O

Some obvious properties of $O_\cdot, F_\cdot$:

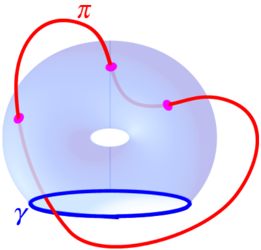
- obstacles O_i are diffeomorphic to the solid tori (like $\mathcal{E}$);
- free spaces F_i are homotopy equivalent to $\mathbb{T}^2$ ("thick torus");
- a loop $\pi : S^1 \to \mathcal{E}$ avoids F *iff* both loops $\pi p_i, i = 1, 2$ in $\mathbb{R}^2$ avoid $\mathcal{D}$.



winding and linking numbers

The winding number of the loop πp_i around $\mathcal{D}$ can be viewed as the *linking* of the loop π in $\mathcal{E}$ with the *equator* in O_i, that is the set of motions in $\mathcal{E}$ taking p_i into a given point in $\mathcal{D}$.

Definition 3 *Let $\pi, \gamma : S^1 \to \mathcal{E}$ be two loops in $\mathcal{E}$, such that γ can be patched by an oriented surface S. Then the intersection number of S and π is called the linking of π and γ.*



Linking number of π and γ is 1.

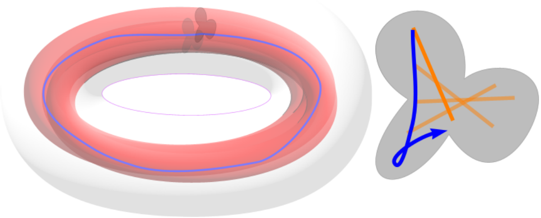
rotating **C** inside $\mathcal{D}$

How to express the notion of "turning **C** around within $\mathcal{D}$"?

rotating C inside $\mathcal{D}$

How to express the notion of "turning **C** around within $\mathcal{D}$"?

- The fact that a loop π in $\mathcal{E}$ is such that both πp_i remain within $\mathcal{D}$ is equivalent to π being within O_{12}.
- The fact that the loop "turns **C** around" means that π goes around the solid torus $\mathcal{E}$ (perhaps, several times).



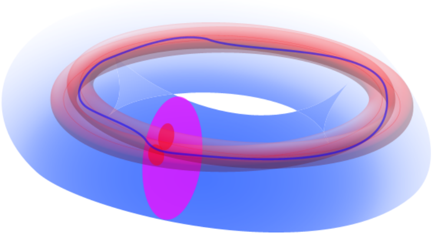
(almost) elementary proof

Now, the “anschaulicher” proof of Proposition 1:

One direction is easy:

If there is a rotation of **C** within $\mathcal{D}$, that is there exists a loop π in O_{12} running around the torus $\mathcal{E}$, then it also runs around $O_i, i = 1, 2$.

Hence, if a loop $\gamma \in F$ has nonzero linking number with O_1, it has nonzero linking number with π, and therefore also nonzero linking number with O_2. Hence, pulling $\mathcal{D}$ through **C** is impossible.



proof, cont'd

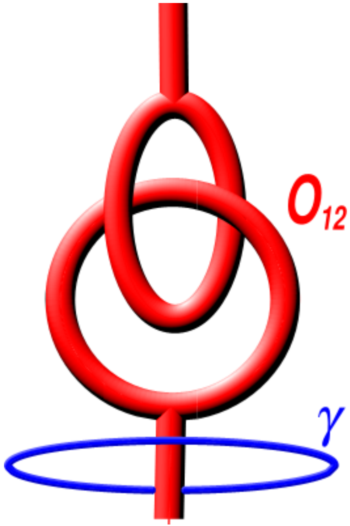
To get the reverse direction, we need a form of *Alexander duality* which in our situation says:

> *The only obstacle to patching a meridian-like loop γ in $F_{12} = \mathcal{E} - O_{12}$ by a 2-chain avoiding O_{12} is an element in $H_1(O_{12})$ representing a nontrivial multiple of $H_1(\mathcal{E})$ (that is by a loop running "around" $\mathcal{E}$).*

Now, if the configuration **C** cannot be rotated within the $\mathcal{D}$, then the large loop γ in $\mathcal{E}$ (obtained by wide circumvention of **C** around $\mathcal{D}$) has linking 1 with both $O_i, i = 1, 2$ and can be patched by a 2-dimensional surface S (with boundary γ).

proof, cont'd

- "Large meridian loop" is an encircling of $\mathcal{D}$ by **C** at distance.



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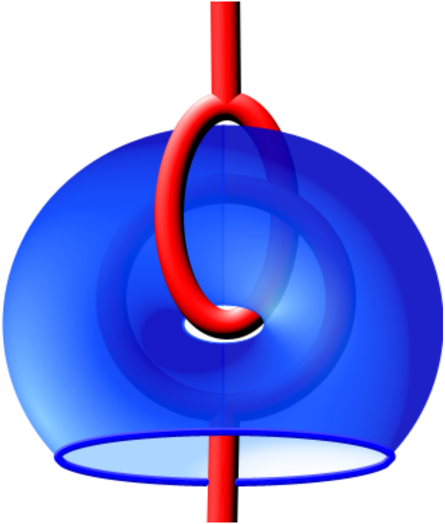
- "Large meridian loop" is an encircling of $\mathcal{D}$ by **C** at distance.
- Impossibility to rotate the two-point configuration **C** within $\mathcal{D}$ implies that this loop is homologically trivial.



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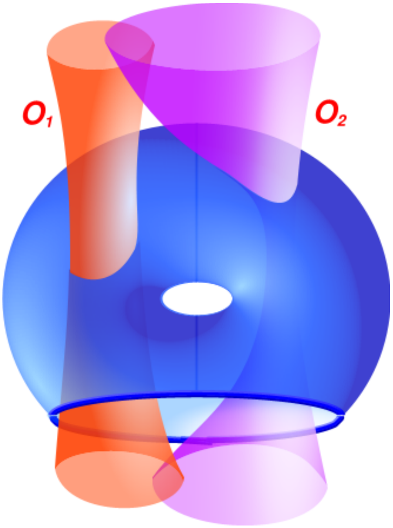

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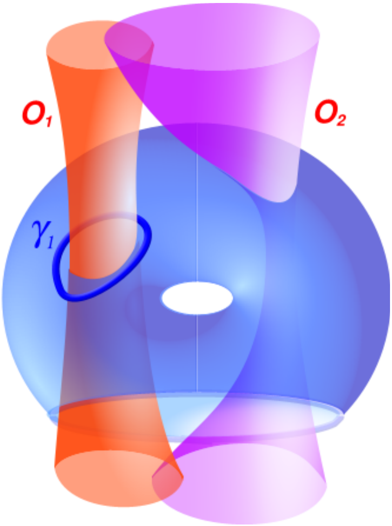
- Impossibility to rotate the two-point configuration **C** within $\mathcal{D}$ implies that this loop is homologically trivial.
- The solid tori O_i intersect S, but not together.
- Hence, one can choose a loop in S encircling O_1 but not O_2. This loop gives the desired pulling.



formal proof

An alternative proof uses the so-called homological Meyer-Vietoris exact sequence:

$$\ldots \xrightarrow{\partial} H_1(\tilde{F}) \xrightarrow{i_1 \oplus i_2} H_1(\tilde{F}_1) \oplus H_1(\tilde{F}_2) \xrightarrow{p} H_1(\tilde{F}_{12}) \xrightarrow{\partial} \ldots$$

(here $\tilde{F}.$'s are the complements to the obstacles not in the solid torus $\mathcal{E}$, but in the 3-sphere $S^3 \supset \mathcal{E}$).

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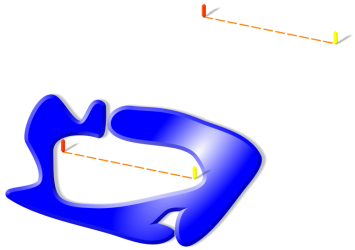
The groups $H_1(\tilde{F}_1), H(\tilde{F}_2)$ and $H_1(\tilde{F}_1 \cap \tilde{F}_2)$ have subgroups generated by the classes λ of large "meridian loops" (in fact, $H_1(\tilde{F}_i)$ are freely generated by these classes). The image of $i_1 \oplus i_2$ contains the diagonal in $H_1(\tilde{F}_1 \cap \tilde{F}_2)$, as $i_1 \oplus i_2 : \lambda \mapsto \lambda \oplus \lambda$. Pulling **C** around $\mathcal{D}$ is possible if and only if the rank of $Im(i_1 \oplus i_2) = 2$, that is if and only if the class of the large meridian loops in $H_1(\tilde{F}_{12})$ vanishes, that is, by Alexander duality, if and only if there is no loop in O_{12} sent to the generator of $H_1(\mathcal{E})$ by the inclusion.

caging

Now we go to our main problem, two-finger caging. We keep assumption that a planar domain $\mathcal{D}$ is homeomorphic to an open disk. We will assume that a reference point configuration $g_* \mathbf{C}, g \in \mathcal{E}$ is far away from $\mathcal{D}$.



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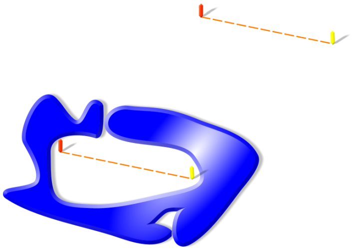

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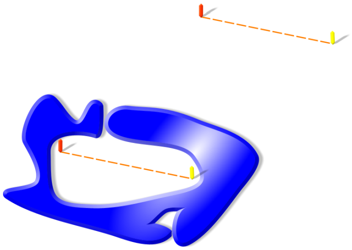

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The configuration e is *not caging* if there exists a path in $\mathcal{E}$ connecting e and g and avoiding O.
In other words, there is no caging, if e and g_* are in the same connected component of $F = \mathcal{E} - O$.

Meyer-Vietoris exact sequence, once again

This means we want to investigate $H_0(F)$.

Consider again the Mayer-Vietoris sequence, but at a different place:

$$\dots \xrightarrow{i_1 \oplus i_2} H_1(F_1) \oplus H_1(F_2) \xrightarrow{p} H_1(F_{12}) \xrightarrow{\partial} H_0(F) \to H_0(F_1) \oplus H_0(F_2) \xrightarrow{\cdots}$$

Proposition 2 *The number of connected components of F not containing g_* equals the rank of the subgroup of $H_1(O_{12})$ having zero linking number with the class of large meridian.*

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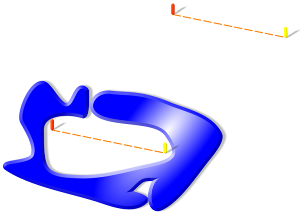
From this viewpoint, the *witnesses* of separation of the components in $\mathcal{E} - O$ are not 2-dimensional surfaces (as one would expect in a 3-dimensional manifold), but 1-dimensional loops.

alternative view

One can associate to two-point configuration **C** in a canonical way a class $\gamma_{\mathbf{C}}$ in $H_1(F_{12})$ (that is, a loop defined up to deformations avoiding O_{12}). The construction is simple:



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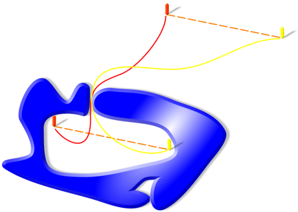

- There exist paths $\gamma_i : I \to \mathcal{E}, i = 1, 2$ connecting e and g_* such that $\gamma_i p_i$ avoids $\mathcal{D}$ at all times (equivalently, the range of γ_i is in F_i.

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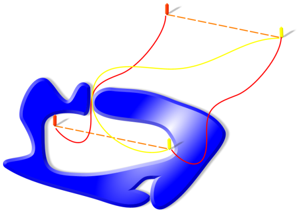

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- Joining these two paths we obtain a loop avoiding O_{12}. This loop (non-unique) defines a class in $H_1(F_{12})$.

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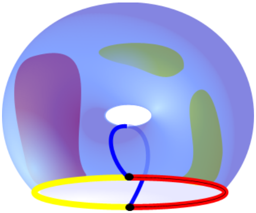

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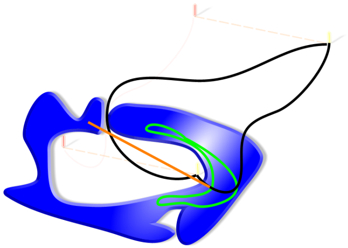
- Joining these two paths we obtain a loop avoiding O_{12}. This loop (non-unique) defines a class in $H_1(F_{12})$.

Proposition 2 implies that if class $\gamma_{\mathbf{C}}$ is trivial, then e and g_* are in the same connected component of F.

Coupling between $H_1(O_{12})$ and $H_1(F)$

The Proposition 2 identifies the obstacles to extracting the 2-finger configuration with loops in O_{12}, that is the loops in $\mathcal{E}$ which keep both points of the configuration within $\mathcal{D}$: a configuration $\mathbf{C}$ is caging if the loop $\gamma_{\mathbf{C}}$ has nontrivial linking with a loop in O_{12}.

To apply this criterion, one needs an efficient algorithm computing the linking number.



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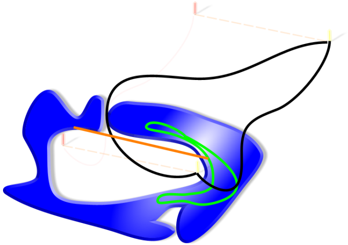

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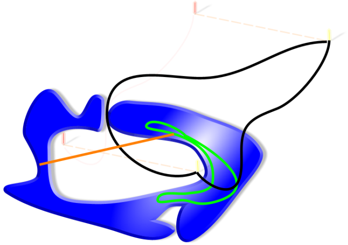

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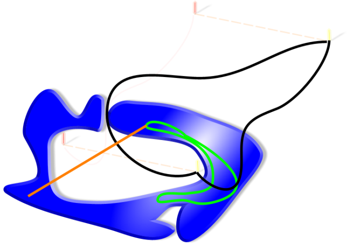

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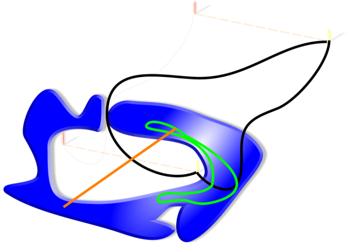

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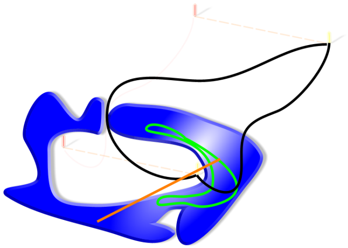

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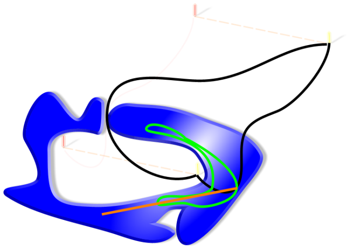

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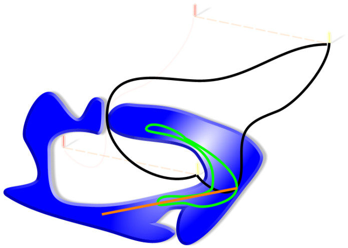

It is more convenient to consider *framed plane curves* in lieu of curves in $\mathcal{E}$.

Computing linking numbers

A framed planar curve is an oriented curve in $\mathbb{R}^2$ with a (continuously varying) unit tangent vector at each point.

A path π in $\mathcal{E}$ and a 2-point configuration $\mathbf{C}$ define a planar framed curve πp_1 framed by the vectors $(\pi p_2 - \pi p_1)/|\pi p_2 - \pi p_1|$.

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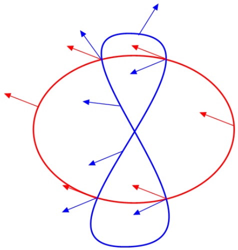

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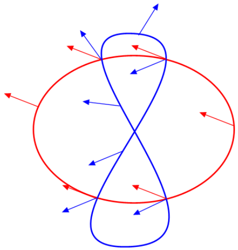

Proposition 3 *The linking number of two plane framed curves is equal to*

$$\sum w_i d_i,$$

where the summation is over the segments of C_2 into which C_1 partitions it, w_i is the winding of the segment, and σ_i is the index of C_1 with respect to this segment.

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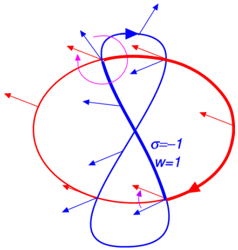

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conclusion

What are the take-aways and future directions:

- Caging problem in 2D turned out to be pleasantly rich with topological content, complementing the traditional paradigms of computational geometry
- The fact that the witnesses for caging are 1-dimensional cycles leads to a novel algorithms of identifying all caging two-finger configurations *(work in progress!)*
- The generalizations to three- and more-finger caging conceptually similar but ask for somewhat more sophisticated machinery, e.g. Meyer-Vietoris spectral sequences *(work in progress!)*

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THANK YOU!