Geodesic flows on hyperbolic orbifolds

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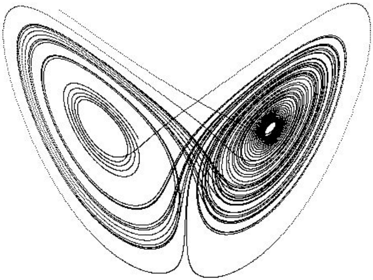
The Lorenz equations

$$\begin{cases} \dot{x} = 10(y - x) \\ \dot{y} = 28x - y - xz \\ \dot{z} = xy - \frac{8}{3}z \end{cases}$$

The Lorenz attractor



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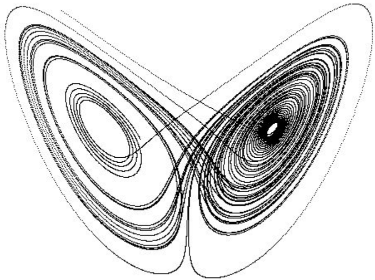
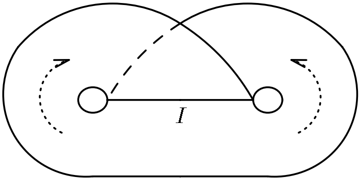
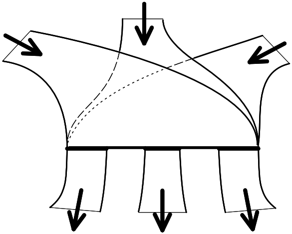




Definition

A *template* is a compact branched two-manifold with boundary and a smooth expanding semiflow built from a finite number of branch line charts.



The template theorem (Birman and Wiliams, 1983)

given a hyperbolic flow ϕ_t on a three-manifold M, the link of all periodic orbits $\mathcal{L}_\phi$ is in bijective correspondence with the link of all periodic orbits $\mathcal{L}_\mathcal{T}$ of a semiflow on an embedded two dimensional branched manifold $\mathcal{T} \subset M$. On any finite sublink, this correspondence is via an ambient isotopy.

Templates are

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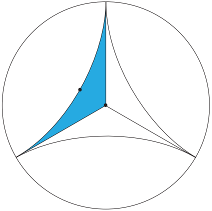
- The only way to topologically analyze periodic orbits
- Extremely hard to compute

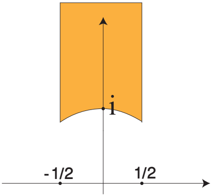
Theorem (Birman and Williams, 1983)

- *There are infinitely many inequivalent Lorenz knots.*
- *Every Lorenz link is a fibered link.*
- *Every Lorenz knot is prime.*
- *Every Lorenz link is a closed positive braid.*
- *Non-trivial Lorenz links have positive signature.*

The modular surface

$M = \mathbb{H}^2 / PSL_2(\mathbb{Z})$





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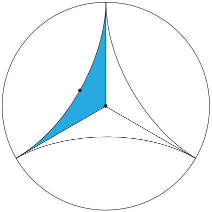
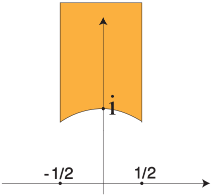
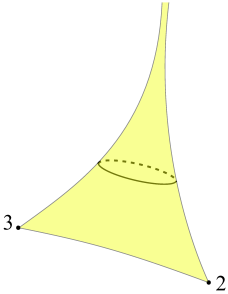
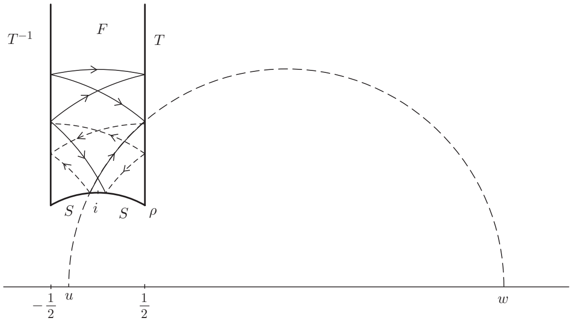






Any point and direction in M define a unique geodesic. This defines a flow ϕ_t, called *the geodesic flow* on $UTM \cong PSL_2(\mathbb{R})/PSL_2(\mathbb{Z}) \cong S^3 \setminus \text{trefoil}$.

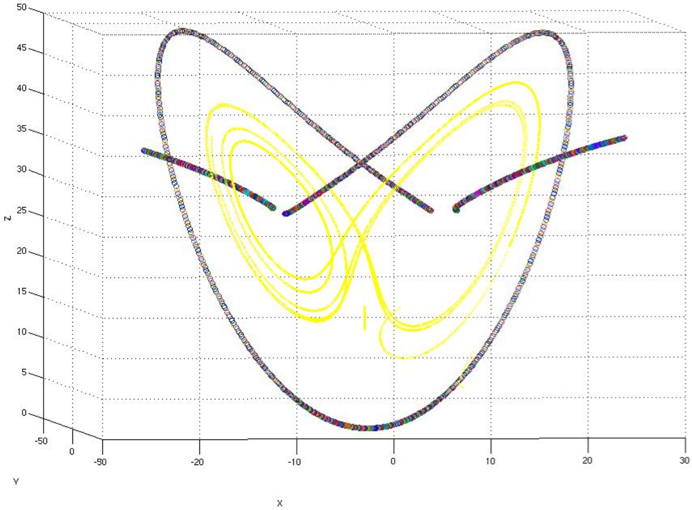


Theorem (Ghys, 2006)

The modular flow has a template identical to the Lorenz template. Thus, isotopy classes of modular and Lorenz knots coincide. Modular links are fibered links and closed positive braids. Nontrivial modular links have nonnegative signature, and modular knots are prime.

Question 1 Why should these knots be identical?

Question 2 Do some properties of the modular knots follow from the fact they are closed geodesics?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-30
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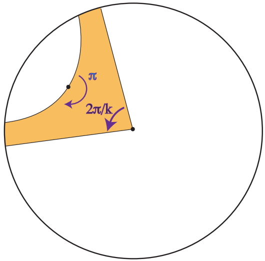
-10
0
10
20
30

Hecke triangle groups

Let $\Gamma_{(2,k)} :=< v, u | v^k = u^2 = e >$ be the $(2, k)$ Hecke triangle group. It has the following representations into $PSL_2(\mathbb{R})$:



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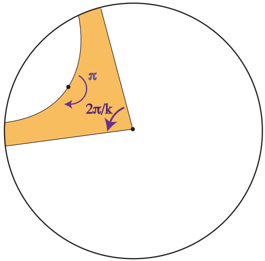

there exists a distance d_0 for which the image of the representation is discrete, yet the orbifold $\Gamma \backslash \mathbb{H}^2$ is of finite volume.

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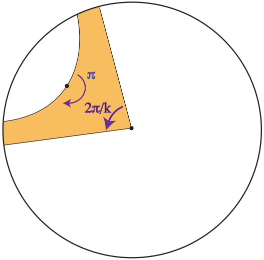
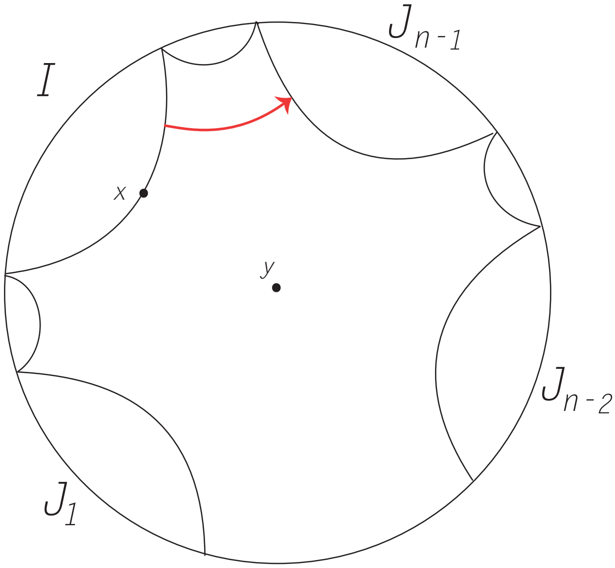
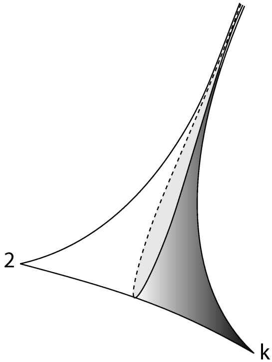


there exists a distance d_0 for which the image of the representation is discrete, yet the orbifold $\Gamma \backslash \mathbb{H}^2$ is of finite volume. Denote the orbifold corresponding to d_0 by $\mathcal{O}_{(2,k)}$.



J_{n-1}
I
x
y
J_{n-2}
J_1

We divide the unit tangent bundle into two parts.



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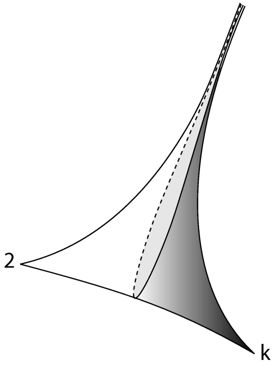

A neighborhood of the 2-cone point is a $(2, 1)$ fibered torus.

We now describe an explicit homeomorphism from pointers (points with attached directions) on the orbifold into the $(2, 1)$ torus.

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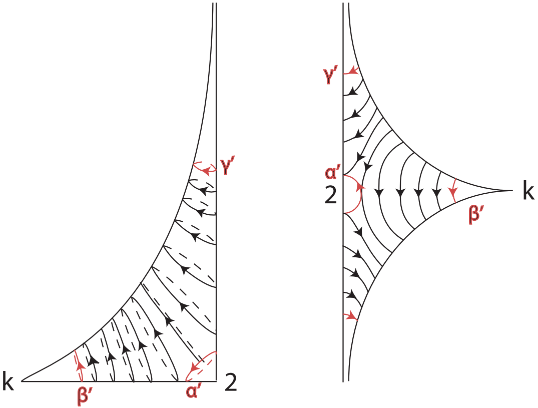
Let A be the orbifold with some small neighborhoods of the cone points removed. Fix a vector field on A as follows.



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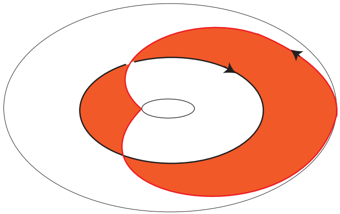
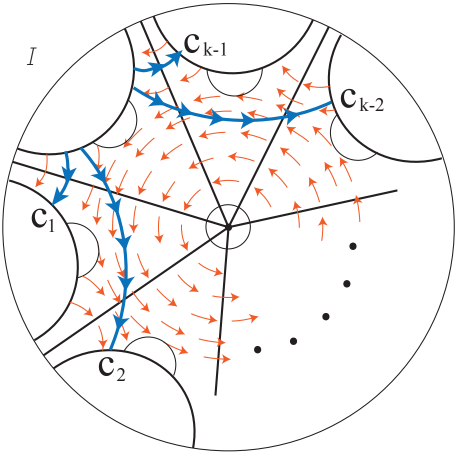
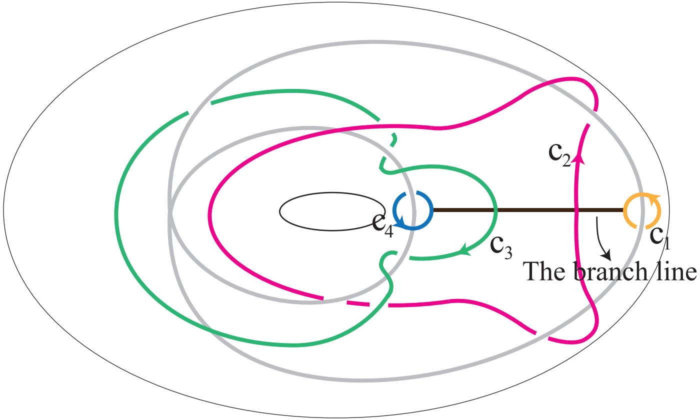


We now use the vector field to draw a template for the geodesic flow on $\mathcal{O}_{(2,k)}$. To this end consider the following figure comparing each of the ear cores to the vector field.

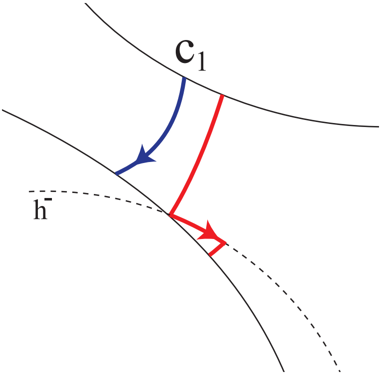


This yields the embedding of the cores into the $(2, 1)$ torus.

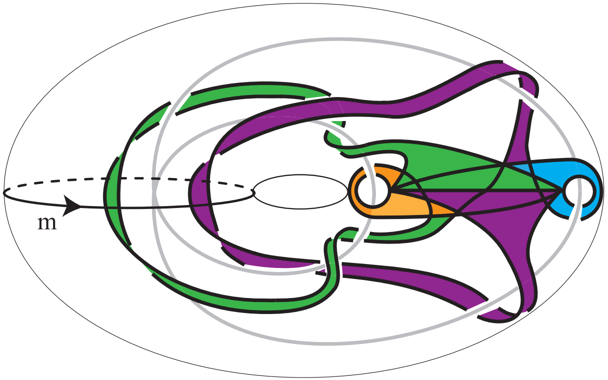


We now draw for each ear one loop in addition to the core, to determine the twists in each of the ears. The second loop will have two parts. One will be a geodesic segment which is not closed, and the other a part of the stable manifold, reaching a geodesic perpendicular to the branch line.

The stable manifold in this setting is just the horocycle corrosponding to the endpoint.



The resulting (2, 5) template



Theorem (TP)

All the templates of the flows on the orbifolds $\mathcal{O}_{(2,k)}$ for odd k, are subtemplates of the following template