Simplicial models for trace spaces

Martin Raussen

Department of Mathematical Sciences
Aalborg University
Denmark

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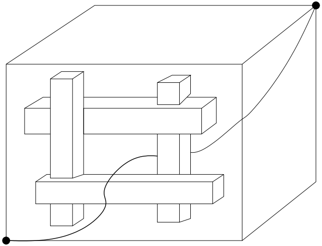
Westfälische
Wilhelms-Universität
Münster

State space and model of trace space

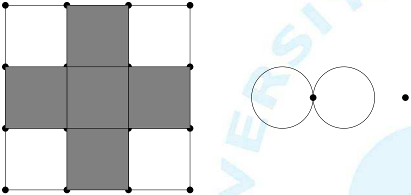
Problem: How are they related?

Example:



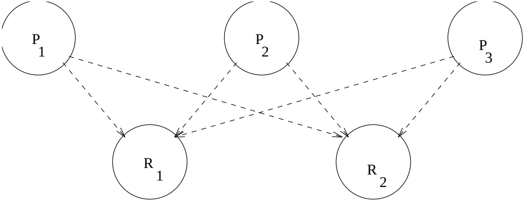
State space X = a cube $\vec{I}^3 \setminus F$ minus 4 box obstructions

$\rightsquigarrow$



Model $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ of trace space contained in a torus $(\partial\Delta^2)^2$ – homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

Mutual exclusion occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

Semaphores!

Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction $P_i : \ldots PR_j \ldots VR_j \ldots$ (E.W. Dijkstra)

P: pakken; V: vrijlaten

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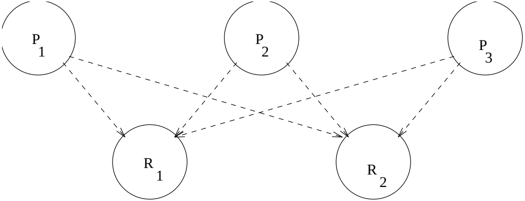

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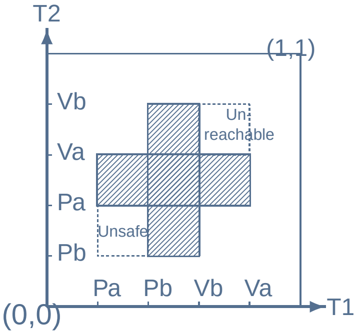
A geometric model: Schedules in "progress graphs"

The Swiss flag example



PV-diagram from

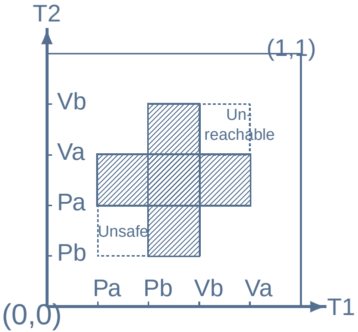
$P_1 : P_a P_b V_b V_a$

$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).

Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

Deadlocks, unsafe and unreachable regions may occur.



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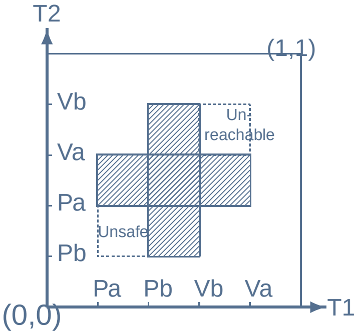

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A linear PV-program can be modelled as the complement of a forbidden region F consisting of a number of holes in an n-cube I^n:

Hole = isothetic hyperrectangle R^i in an n-cube.

State space

$X = \vec{I}^n \setminus F, \; F = \bigcup_{i=1}^{l} R^i, \; R^i =]a_1^i, b_1^i[\times \cdots \times]a_n^i, b_n^i[.$

with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.

X inherits a partial order from $\vec{I}^n$.

More general PV-programs:

- Replace $\vec{I}^n$ by a product $\Gamma_1 \times \cdots \times \Gamma_n$ of digraphs.
- Holes have then the form $p_1^i((0,1)) \times \cdots \times p_n^i((0,1))$ with $p_j^i : \vec{I} \to \Gamma_j$ a directed injective (d-)path.
- Pre-cubical complexes: like pre-simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.

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Main interest: Spaces of d-paths/traces – up to dihomotopy

- X a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving")
- $\vec{P}(X)(a,b) = \{p : \vec{I} \to X \mid p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a,b) = \vec{P}(X)(a,b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a,b) \simeq \vec{T}(X)(a,b)$.
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Aim: Explicit description of the homotopy type of $\vec{P}(X)(a,b)$; in particular of its path components, ie the dihomotopy classes of d-paths.
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Covers of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

by contractible or empty subspaces

$X = \vec{I}^n \setminus F, F = \bigcup_{i=1}^{l} R^i; R^i = [\mathbf{a}^i, \mathbf{b}^i]; \mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition

$$X_{j_1,\dots,j_l} = \{x \in X \mid \forall i : x_{j_i} \le a^i_{j_i} \vee \exists k : x_k \ge b^i_k\}$$
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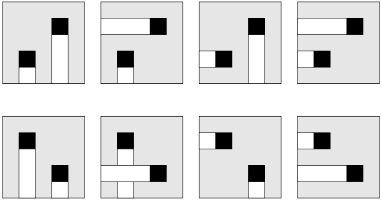
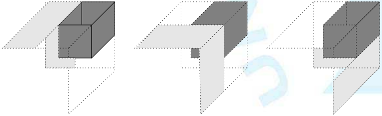
Examples:



3D:



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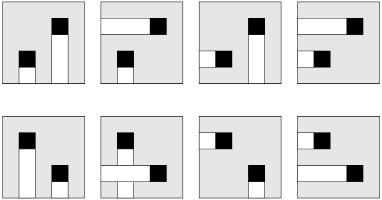

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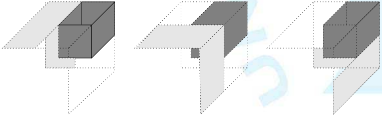

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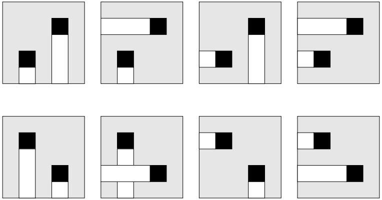

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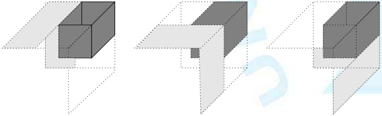

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More intricate subspaces as intersections

either empty or contractible

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$$X_{J_1,\dots,J_l} = \bigcap_{j_i \in J_i} X_{j_1,\dots,j_l}; \qquad J_1,\dots,J_l \subseteq [1:n]$$
$$= \{x \in X \mid \forall i, j_i \in J_i : x \leq \mathbf{b}^i \Rightarrow x_{j_i} \leq a^i_{j_i}\}$$

Theorem

$\emptyset \neq J_1,\dots,J_l \subseteq [1:n] \Rightarrow$
$\vec{P}(X_{J_1,\dots,J_l})(\mathbf{0},\mathbf{1})$ *is either empty or contractible.*

Proof.

relies on: Subspaces $X_{J_1,\dots,J_l}$ are closed under $\vee$ = l.u.b. □

Question: For which $J_1,\dots,J_l \subseteq [1:n]$ is

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$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices

$M^R_{l,n}$ no row vector is the zero vector

$M^C_{l,n}$ every column vector is a unit vector

Restriction to Index sets	$\leftrightarrow$	Matrix sets
$(\mathcal{P}([1:n]))^l$	$\leftrightarrow$	$M_{l,n}$
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Poset category – Combinatorics

$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M^R_{l,n} \subseteq M_{l,n}$

$J \leftrightarrow M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

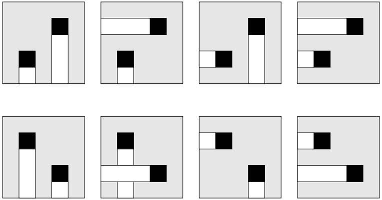
Prodsimplicial complex – Topology

$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$

$\Delta^{|J_1|-1}_{J_1} \times \cdots \times \Delta^{|J_l|-1}_{J_l} \subseteq \mathbf{T}(X)(\mathbf{0},\mathbf{1})$

$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$

Examples:



- $\mathbf{T}(X_1)(\mathbf{0},\mathbf{1}) = (\partial\Delta^1)^2 = 4*$
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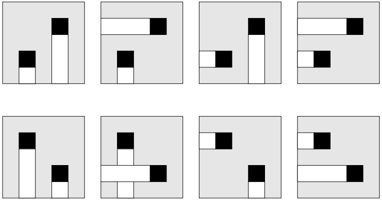

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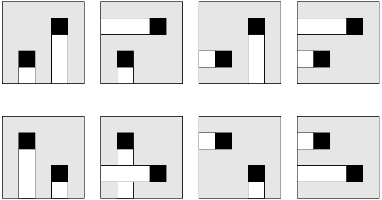

- $\mathbf{T}(X_1)(\mathbf{0},\mathbf{1}) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2)(\mathbf{0},\mathbf{1}) = 3*$

$$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} \supseteq \mathcal{C}(X)(\mathbf{0},\mathbf{1})$$

Further examples

(1) $X = \vec{I}^n \setminus \vec{J}^n$

- $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^R_{1,n} \setminus \{[1,\dots 1]\}$.
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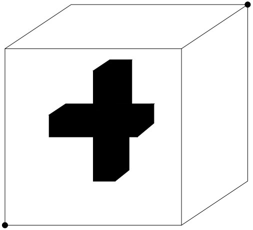
(2)



- $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\}$.
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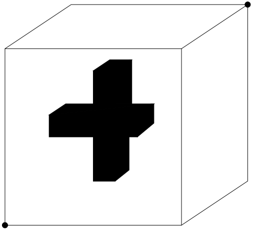

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Homotopy equivalence between trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ and prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(J_1,\dots,J_l) = \vec{P}(X_{J_1,\dots,J_l})(\mathbf{0},\mathbf{1})$,
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- colim $\mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, colim $\mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
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- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
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From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of trace space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

- Is $\vec{T}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected? Other topological properties?

The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces.
There are fast algorithms to calculate the homology groups of these chain complexes even for very big complexes.
Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0},\mathbf{1}))$.
For path-components, there might be faster "discrete" methods.
Even if "exponential explosion" prevents precise calculations, inductive determination (round by round) of general properties ((simple) connectivity) may be possible.
Implementation in ALCOOL: progress at CEA/LIX-lab.

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Deadlocks and unsafe regions determine $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$

A dual view: **extended** hyperrectangles R^i_j
$:= [0,b^i_1[\times\cdots\times[0,b^i_{j-1}[\times]a^i_j,b^i_j[\times[0,b^i_{j+1}[\times\cdots\times[0,b^i_n[\supset R^i$.

$$X_M = X \setminus \bigcup_{m_{ij}=1} R^i_j.$$

Theorem

The following are equivalent:

1. $\vec{P}(X_M)(\mathbf{0},\mathbf{1})=\emptyset \Leftrightarrow M\not\in\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
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3. *Mere combinatorics: Checking a bunch of inequalities: There is a map $i:[1:n]\to[1:l]$ such that $a^{i(j)}_j < b^{i(k)}_j$ for all $1\le j,k\le n$.*

OBS: $\chi(\text{graph}(i)) = M(i)\in M^C_{l,n}$!

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Which of the l^n matrices in $M_{l,n}^C$ belong to $D(X)(\mathbf{0},\mathbf{1})$?

A matrix $M = M(i) \in M_{l,n}^C$ is described by a (choice) map

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Partial orders and order ideals on matrix spaces
and an order preserving map Ψ

Consider $\Psi : M_{l,n} \to \mathbf{Z}/2,\ \Psi(M) = 1 \Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$.

- Ψ is order preserving, in particular: $\Psi^{-1}(0), \Psi^{-1}(1)$ are closed in opposite senses: $M \leq N : \Psi(N) = 0 \Rightarrow \Psi(M) = 0; \Psi(M) = 1 \Rightarrow \Psi(N) = 1$ (thus $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ prodsimplicial).
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Algorithmics: Construct $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})$ incrementally (checking for one matrix $N \in D(X)(\mathbf{0},\mathbf{1})$ at a time), starting with matrix $\mathbf{1}$:

1. $N_{i+1} \not\leq M \in \mathcal{C}^i(X) \Rightarrow M \in \mathcal{C}^{i+1}(X)$;
2. $N_{i+1} \leq M \Rightarrow M$ is replaced by n matrices M^j with one additional 0. Example: $X = \vec{I}^n \setminus \vec{J}^n$.

A matrix in $D(X)(\mathbf{0},\mathbf{1})$ describes a hyperedge on the vertex set $[1:l] \times [1:n]$; $D(X)(\mathbf{0},\mathbf{1})$ describes a hypergraph.

A transversal in a hypergraph is a vertex set that has non-empty intersection with each hyperedge

$\leftrightarrow$ a matrix L such that $\forall N \in D(X)(\mathbf{0},\mathbf{1})\ \exists (i,j) : l_{ij} = n_{ij} = 1$.

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Conclusion: Search for matrices in $A_{max}(\mathbf{0},\mathbf{1})$ corresponds to search for minmal transversals in $D(X)(\mathbf{0},\mathbf{1})$.

In our case: All hyperedges have same cardinality n, include one element per column.

Algorithmics: Construct $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})$ incrementally (checking for one matrix $N \in D(X)(\mathbf{0},\mathbf{1})$ at a time), starting with matrix $\mathbf{1}$:

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- General end points: $\vec{P}(X)(\mathbf{c},\mathbf{d})$; iterative calculations; relations...
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Products of digraphs instead of $\vec{I}^n$:
$\Gamma = \prod_{j=1}^{n} \Gamma_j$, state space $X = \Gamma \setminus F$,
F a product of generalized hyperrectangles R^i.

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Represent a path component $C \in \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$ by (regular) d-paths $p_j \in \vec{P}(\Gamma_j)(x_j, y_j)$ – an interleaving.
The map $c : \vec{I}^n \to \Gamma, c(t_1, \dots, t_n) = (c_1(t_1), \dots, c_n(t_n))$ induces a homeomorphism $\circ c : \vec{P}(\vec{I}^n)(\mathbf{0}, \mathbf{1}) \to C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.

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Pull back F via c:
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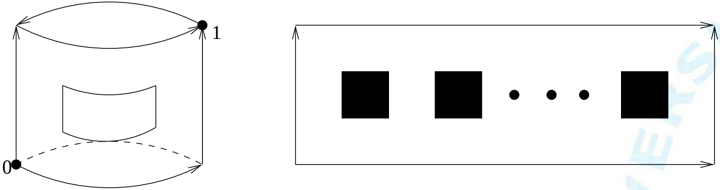

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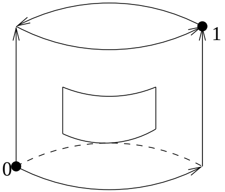

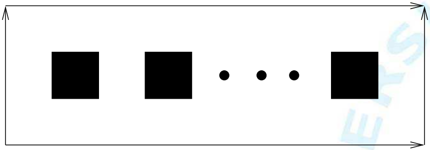

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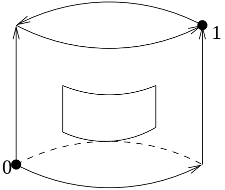

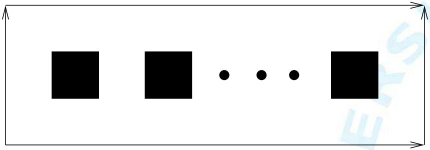

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Thank you for your attention!

Want to know more?

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